2

Linear Programming Problem

Contents

2.1 Introduction

- Many business and economic activities are concerned with the problem of planning. If the supply of the resources are unlimited, the need for linear programming problems would not arise at all. In each case, limited resources are available, and to make use of this resource, programming and planning problems could be formulated to maximise/minimise a linear form of profit/cost function whose variables are restricted to values satisfying a system of linear constraints (a set of linear equations/or inequations). The term 'programming' refers to the process of determining a plan of action. Linear programming is a technique for determining an optimum schedule of independent activities in view of the available resources. Linear relationship between two or more variables is the one in which the variables are directly or precisely proportional.
- Linear programming is perhaps the most widely applied mathematical technique that helps managers in decision-making and planning for the optimal allocation of limited resources. It deals with the optimization (maximization or minimization) of a function of variables known as *objective function*, subject to a set of linear equations and/or inequalities known as *constraints.* The objective function may be profit, cost, production capacity or any other measure of effectiveness, which is to be obtained in the best possible or optimal manner. The constraints may be imposed by different resources such as raw material availability, market demand, production process and equipment, storage capacity, etc. By linearity is meant a mathematical expression in which the expressions among the variables are linear *e.g.*, the expression $a_x x_x + a_2 x_2 + a_3 x_3 + ... + a_n x_n$ is linear. Higher powers of the variables or their products donot appear in the expressions for the objective function as well as the constraints (they donot have expressions like x_1^3, $x_2^{3/2}$, $x_1 x_2$, $a_1 x_1 + a_2 \log x_2$, etc.). The variables obey the properties of *proportionality* (*e.g.*, if a product requires 3 hours of machining time, 5 units of it will require 15 hours) and *additivity* (*e.g.*, amount of a resource required for a certain number of products is equal to the sum of the resource required for each).
- It was in 1947 that George Dantzig and his associates found out a technique for solving military planning problems while they were working on a project for U.S. Air Force. This technique consisted of representing the various activities of an organization as a linear programming (L.P.) model and arriving at the optimal programme by minimizing a linear objective function. Afterwards, Dantzig suggested this approach for solving business and industrial problems. He also developed the most powerful mathematical tool known as "simplex method" to solve linear programming problems.

2.2 Requirements for A Linear Programming Problem

- All organizations, big or small, have at their disposal, men, machines, money and materials, the supply of which may be limited. If the supply of these resources were unlimited, the need for management tools like linear programming would not arise at all. Supply of resources being limited, the management must find the best allocation of its resources in order to maximize the profit or minimize the cost/loss or utilize the production capacity to the maximum extent. However, this involves a number of problems which can be overcome by quantitative methods, particularly the linear programming.
- Generally speaking, linear programming can be used for optimization problems if the following conditions are satisfied:
  1. There must be a well defined objective function (profit, cost or quantities produced) which is to be either maximized or minimized and which can be expressed as a linear function of decision variables.
  2. There must be constraints on the amount or extent of attainment of the objective and these constraints must be capable of being expressed as linear equations or inequalities in terms of variables.

3. There must be alternative courses of action. For example, a given product may be processed by two different machines and problem may be as to how much of the product to allocate to which machine.
4. Another necessary requirement is that decision variables should be interrelated and nonnegative. The non-negativity condition shows that linear programming deals with real life situations for which negative quantities are generally illogical.
5. As stated earlier, the resources must be in limited supply. For example, if a firm starts producing greater number of a particular product, it must make smaller number of other products as the total production capacity is limited.

2.3 Assumptions In Linear Programming Models

- A linear programming model is based on the following assumptions:

1. **Proportionality**
A basic assumption of linear programming is that proportionality exists in the objective function and the constraints. This assumption implies that if a product yields a profit of ₹ 100, the profit earned from the sale of 15 such products will be ₹ (100 x 15) = ₹ 1500. This may not always be true because of quantity discounts. Further, even if the sale price is constant, the manufacturing cost may vary with the number of units produced and so may vary the profit per unit. Likewise, it is assumed that if one product requires processing time of 2 hours, then ten such products will require processing time of 2 x 10 = 20 hours. This may also not be true as the processing time per unit often decreases with increase in number of units produced. The real world situations may not be strictly linear. However, assumed linearity represents their close approximations and provides very useful answers.
2. **Additivity**
It means that if we use 0 hours on machine A to make product 1 and t2 hours to make product 2, the total time required to make products 1 and 2 on machine A is tx + t2 hours. This, however, is true only if the change-over time from product 1 to product 2 is negligible. Some processes may not behave in this way. For example, when several liquids of different chemical compositions are mixed, the resulting volume may not be equal to the sum of the volumes of the individual liquids. 3.
3. **Continuity**
Another assumption underlying the linear programming model is that the decision variables are continuous i.e., they are permitted to take any non-negative values that satisfy the constraints. However, there are problems wherein variables are restricted to have integral values only. Though such problems, strictly speaking, are not linear programming problems, they are frequently solved by linear programming techniques and the values are then rounded off to nearest integers to satisfy the constraints. This approximation, however, is valid only if the variables have large optimal values. Further, it must be ascertained whether the solution represented by the rounded values is a feasible solution and also whether the solution is the best integer solution.
4. **Certainty**
Another assumption underlying a linear programming model is that the various parameters, namely, the objective function coefficients, R.H.S. coefficients of the constraints and resource values in the constraints are certainly and precisely known and that their values do not change with time. Thus the profit or cost per unit of the product, labour and materials required per unit, availability of labour and materials, market demand of the product produced, etc. are assumed to be known with certainty. The linear programming problem is, therefore, assumed to be deterministic in nature.
5. **Finite Choices**

A linear programming model also assumes that a finite (limited) number of choices (alternatives) are available to the decision-maker and that the decision variables are interrelated and non negative. The non-negativity condition shows that linear programming deals with real-life situations as it is not possible to produce/use negative quantities. Mathematically these non-negativity conditions do not differ from other constraints. However, since while solving the problems they are handled differently from the other constraints, they are termed as non-negativity restrictions and the term constraints is used to represent constraints other than non-negativity restrictions.

2.4 Applications of Linear Programming Method

- Though, in the world we live, most of the events are non-linear, yet there are many instances of linear events that occur in day-to-day life. Therefore, an understanding of linear programming and its application in solving problems is utmost essential for today's managers.
- Linear programming techniques are widely used to solve a number of business, industrial, military, economic, marketing, distribution and advertising problems. Three primary reasons for its wide use are:
  1. A large number of problems from different fields can be represented or at least approximated to linear programming problems.
  2. Powerful and efficient techniques for solving L.P. problems are available.
  3. L.P models can handle data variation (sensitivity analysis) easily.
- However, solution procedures are generally iterative and even medium size problems require manipulation of large amount of data. But with the development of digital computers, this disadvantage has been completely overcome as these computers can handle even large L.P. problems in comparatively very little time at a low cost.

2.5 Areas of Application Of Linear Programming

- Linear programming is one of the most widely applied techniques of operations research in business, industry and numerous other fields. A few areas of its application are given below.

2.5.1 Industrial Applications

a) Product mix problems

An industrial organisation has available a certain production capacity (men, machines, money, materials, market, etc.) on various manufacturing processes to manufacture various products. Typically, differents products will have different selling prices, will require different amounts of production capacity at the several processes and will therefore, have different unit profits; there may also be stipulations (conditions) on maximum and/or minimum product levels. The problem is to determine the product mix that will maximize the total profit.

b) Blending problems

These problems are likely to arise when a product can be made from a variety of available raw materials of various compositions and prices. The manufacturing process involves blending (mixing) some of these materials in varying quantities to make a product of the desired specifications.

For instance, different grades of gasoline are required for aviation purposes. Prices and specifications such as octane ratings, tetra ethyl lead concentrations, maximum vapour pressure, etc. of input ingredients are given and the problem is to decide the proportions of these ingredients to make the desired grades of gasoline so that (/) maximum output is obtained and (//) storage capacity restrictions are satisfied. Many similar situations such as preparation of different kinds of whisky, chemicals, fertilisers and alloys, etc. have been handled by this technique of linear programming.

c) **Production scheduling problems**
They involve the determination of optimum production schedule to meet fluctuating demand. The objective is to meet demand, keeping inventory and employment at reasonable minimum levels, while minimizing the total cost of production and inventory.

d) **Trim loss problems**
They are applicable to paper, sheet metal and glass manufacturing industries where items of standard sizes have to be cut to smaller sizes as per customer requirements with the objective of minimizing the waste produced.

e) **Assembly-line balancing**
It relates to a category of problems wherein the final product has a number of different components assembled together. These components are to be assembled in a specific sequence or set of sequences. Each assembly operator is to be assigned the task / combination of tasks so that his task time is less than or equal to the cycle time.

f) **Make-or-buy (sub-contracting) problems**
They arise in an organisation in the face of production capacity limitations and sudden spurt in demand of its products. The manufacturer, not being sure of the demand pattern, is usually reluctant to add additional capacity and has to make a decision regarding the products to be manufacured with his own resources and the products to be sub-contracted so that the total cost is minimized.

2.5.2 Management Applications

a) **Media selection problems**
They involve the selection of advertising mix among different advertising media such as T.V., radio, magazines and newspapers that will maximize public exposure to company's product. The constraints may be on the total advertising budget, maximum expenditure in each media, maximum number of insertions in each media and the like.

b) **Portfolio selection problems**
They are frequently encountered by banks, financial companies, insurance companies, investment services, etc. A given amount is to be allocated among several investment alternatives such as bonds, saving certificates, common stock, mutual fund, real estate, etc. to maximize the expected return or minimize the expected risk.

c) **Profit planning problems**
They involve planning profits on fiscal year basis to maximize profit margin from investment in plant facilities, machinery, inventory and cash on hand.

d) **Transportation problems**
They involve transportation o f products from, say, n sources situated at different locations to, say, m different destinations. Supply position at the sources, demand at destinations, freight charges and storage costs, etc. are known and the problem is to design the optimum transportation plan that minimizes the total transportation cost (or distance or time).

e) **Assignment problems**
They are concerned with allocation of facilities (men or machines) to jobs. Time required by each facility to perform each job is given and the problem is to find the optimum allocation (one job to one facility) so that the total time to perform the jobs is minimized.

f) **Manpower scheduling problems**
They are faced by big hospitals, restaurants and companies operating in a number of shifts. The problem is to allocate optimum man-power in each shift so that the overtime cost is minimized.

2.5.3 Miscellaneous Applications

a) Diet problems
They form another important category to which linear programming has been applied. Nutrient contents such as vitamins, proteins, fats, carbohydrates, starch, etc. in each of a number of food stuffs is known. Also the minimum daily requirement of each nutrient in the diet as well as the cost of each type of food stuff is given and the problem is to determine the minimum cost diet that satisfies the minimum daily requirement of nutrients.

b) Agriculture problems
These problems are concerned with the allocation of input resources such as acreage of land, water, labour, fertilisers and capital to various crops so as to maximize net revenue.

c) Flight scheduling problems
They are devoted to the determination of the most economical patterns and timings of flights that result in the most efficient use of aircrafts and crews.

d) Environment protection
They involve analysis of different alternatives for efficient waste disposal, paper recycling and energy policies.

e) Facilities location
These problems are concerned with the determination of best location of public parks, libraries and recreation areas, hospital ambulance depots, telephone exchanges, nuclear power plants, etc.

2.6 Mathematical Formulation of Linear Programming Problem

- The procedure for mathematical formulation of LPP consists of the following steps:
  - **Step 1:** Identify the decision variables of the problem.
  - **Step 2:** Formulate the objective function to be optimised (maximised or minimised) as a linear function of the decision variables.
  - **Step 3:** Formulate the constraints of the problem such as resource limitations, market conditions, interrelation between variables and others as linear equation or inequations in terms of the decision variables.
  - **Step 4:** Add the non-negativity constraint so that negative values of the decisions variables do not have any valid physical interpretation.
- The objective function, the set of constraint and the non-negative constraint together form a linear programming problem.

2.7 Graphical Method of Solution

- Once a problem is formulated as mathematical model, the next step is to solve the problem to get the optimal solution. A linear programming problem with only two variables presents a simple case, for which the solution can be derived using a graphical or geometrical method. Though, in actual practice such small problems are rarely encountered, the graphical method provides a pictorial representation of the solution process and a great deal of insight into the basic concepts used in solving large L.R problems. This method consists of the following steps:
  1. Represent the given problem in mathematical form i.e., formulate the mathematical model for the given problem.
  2. Draw the x_1 and x_2 axis. The non-negativity restrictions $x_1 \geq 0$ and $x_2 \geq 0$ imply that the values of the variables x_1 and x_2 can lie only in the first quadrant. This eliminates a number of infeasible alternatives that lie in 2^{nd}, 3^{rd} and 4^{th} quadrants.

3. Plot each of the constraint on the graph. The constraints, whether equations or inequalities are plotted as equations. For each constraint, assign any arbitrary value to one variable and get the value of the other variable. Similarly, assign another arbitrary value to the other variable and find the value of the first variable. Plot these two points and connect them by a straight line. Thus each constraint is plotted as line in the first quadrant.
4. Identify the feasible region (or solution space) that satisfies all the constraints simultaneously. For $\geq$ type constraint, the area on or above the constraint line i.e., away from the origin and for $\leq$ type constraint, the area on or below the constraint line i.e., towards origin will be considered. The area common to all the constraints is called feasible region and is shown shaded. Any point on or within the shaded region represents a feasible solution to the given problem. Though a number of infeasible points are eleminated, the feasible region still contains a large number of feasible points. Feasible region is also called convex polygen. Which is a convex set formed by the intersection of finite number of closed half-planes.
5. Use iso-profit (cost) function line approach. For this, plot the objective function by assuming Z = 0. This will be a line passing through the origin. As the value of Z is increased from zero, the line starts moving to the right, parallel to itself. Draw lines parallel to this line till the line is farthest way from the origin (for a maximization problem). For a minimization problem, the line will be nearest to the origin. The point of the feasible region through which this line passes will be the optimal point. It is possible that this line may coincide with one of the edges of the feasible region. In that case, every point on that edge will give the same maximum/minimum value of the objective function and will be the optimal point.

   Alternatively use extrema point enumeration approach. For this, find the co-ordinates of each extreme point (or corner point or vertex) of the feasible region. Find the value of the objective function at each extreme point. The point at which objective function is maximum/minimum is the optimal point and its co-ordinates give the optimal solution.

2.8 The General Linear Programming Problem

- The general linear programming problem can be expressed as follows:
- Find the values of variables $x_1, x_2, \ldots, x_n$ which maximize (or minimize) an objective function which is a linear function of variables, such as

$$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \qquad \textit{Eq. (2.1)}$$

subject to the constraints

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ (\leq, =, \geq)\ b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \ (\leq, =, \geq)\ b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \ (\leq, =, \geq)\ b_m \end{aligned}\right\} \qquad \textit{Eq. (2.2)}$$

and meet the non-negativity restrictions

$$x_1, x_2, \ldots, x_n \geq 0 \qquad \textit{Eq. (2.3)}$$

- For each constraint one and only one of signs $(\leq, =, \geq)$ holds but the sign may vary from one constraint to another.

2.9 Theory of Simplex Method

- Simplex method, also called simplex technique or simplex algorithm was developed in 1947 by G.B. Dantzig, an American mathematician. It has the advantage of being universal, i.e., any linear model for which the solution exists, can be solved by it. In principle, it consists of starting with a certain solution of which all that we know is that it is basic feasible, i.e., it satisfies the constraints as well as non-negativity conditions ($x_j \geq 0, j = 1,2,3,\ldots,n$). We, then, improve upon this solution at consecutive stages, until, after a certain finite number of stages, we arrive at the optimal solution. The method also helps the decision-maker to identify the redundant constraints, an unbounded solution, multiple solutions and an infeasible solution.

- The simplex method provides an algorithm which consists in moving from one vertex of the region of feasible solutions to another in such a manner that the value of the objective function at the succeeding vertex is less in a minimization problem (or more in a maximization problem) than at the preceding vertex. This procedure of jumping from one vertex to another is then repeated. Since the number of vertices is finite, this method leads to an optimal vertex in a finite number of steps. The basis of the simplex method consists of two fundamental conditions:
    1. **The feasibility condition:** It ensures that if the starting solution is basic feasible, only basic feasible solutions will be obtained during computation.
    2. **The optimality condition:** It guarantees that only better solutions (as compared to the current solution) will be encountered.

2.10 Some Important Definitions

- Consider the general linear programming problem involving n variables and m constraints ($m \leq n$):
- Determine the values of variables $x_1, x_2, \ldots, x_n$ which

maximize

$$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2, \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m, \end{aligned}$$

where $x_1, x_2, \ldots x_n \geq 0$.

- Introducing slack variables $s_1, s_2, \ldots s_n$ in the constraints, it can be put in the following standard form :

maximize

$$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \qquad \textit{Eq. (2.4)}$$

subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + s_1 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + s_2 &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + s_m &= b_m \end{aligned} \qquad \textit{Eq. (2.5)}$$

where

$$x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m \geq 0 \qquad \textit{Eq. (2.6)}$$

1. **Solution**
   A set of variables $[x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m]$ is called a solution to L.P. problem if it satisfies the constraints (2.5).
2. **Feasible solution**
   A set of variables $[x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m]$ is called a feasible solution to L.P. problem if it satisfies the constraints (2.5) as well as non-negativity restrictions (2.6).
3. **Basic solution**
   A solution obtained by setting any n variables (among m + n variables) equal to zero and solving for remaining m variables (provided the determinant of the coefficients of these m variables is non-zero)

is called a basic solution. These m variables (some of them may be zero) are called basic variables and the remaining n variables that have been put equal to zero each are called non-basic variables.

4. **Basic feasible solution**
   It is a basic solution that also satisfies the non-negativity restrictions (2.6). All variables in a basic feasible solution are ≥ 0. Every basic feasible solution of a problem is an extreme point of the convex set of feasible solutions and every extreme point is a basic feasible solution of the set of constraints.
5. **Non-degenerate basic feasible solution**
   It is a basic feasible solution in which all the m basic variables are positive (> 0) and the remaining n variables are zero each.
6. **Degenerate basic feasible solution**
   It is a basic feasible solution in which one or more of the m basic variables are equal to zero.
7. **Optimal basic feasible solution**
   It is the basic feasible solution that also optimizes the objective function (2.4).
8. **Unbounded solution**
   If the value of the objective function can be increased or decreased indefinitely, the solution is called unbounded solution.

2.11 The Simplex Method / Technique / Algorithm

- The graphical method cannot be applied when the number of variables involved in the L.P. problem is more than three or rather two, since even with three variables the graphical solution becomes tedius as it involves intersection of planes in three dimensions. The simplex method, developed by Prof. George B. Dantzig, can be used to solve any L.P. problem (for which the solution exists) involving any number of variables and constraints (hundreds or even thousands).
- The computational procedure in the simplex method is based on the fundamental property that the optimal solution to an L.P. problem , if it exists, occurs only at one of the corner points of the feasible region. The simplex method always starts with initial basic feasible solution i.e., origin, which is one of the corner points of the feasible region. This solution is then tested i.e., it is ascertained whether improvement in the value of the objective function is possible by moving to the next comer point of the feasible region. If so, the solution at this point is obtained. This search for better comer point is repeated, till after a finite number of trials, the optimal solution, if it exists, is obtained.

2.12 Artificial Variables Techniques

- The problems in which constraints were of $(\leq)$ type (with non-negative right-hand sides). The introduction of slack variables readily provided the initial basic feasible solution. There are, however, many linear programming problems where slack variables cannot provide such a solution. In these problems at least one of the constraints is of $(\geq)$ or $(=)$ type.
- In such cases, we introduce another type of variables called artificial variables. These variables are fictitious and have no physical meaning. They assume the role of slack variables in the first iteration, only to be replaced at a later iteration. Thus they are merely a device to get the starting basic feasible solution so that simplex algorithm can be applied as usual to get optimal solution.
- There are two (closely related) techniques available to solve such problems. They are
  1. **The big M-method** or M-technique or method of penalties due to A. Chames.
  2. **The two-phase method** due to Dantzig, Orden and Wolfe.

2.12.1 The Big M-Method

- This method consists of the following basic steps :

- **Step 1.** Express the linear programming problem in standard form by introducing slack variables. These variables are added to the left-hand sides of the constraints of $(\leq)$ type and subtracted from the constraints of $(\geq)$ type.
- **Step 2.** Add non-negative variables to the left-hand sides of all the constraints of initially $(\geq)$ or $(=)$ type. These variables are called artificial variables. The purpose of introducing the artificial variables is just to obtain an initial basic feasible solution. They have, however, two drawbacks:
    1. They are fictitious, have no physical meaning or economic significance and have no relevance to the problem.
    2. Their introduction (addition) violates the equality of constraints that has been already established in step 1.
- They are, therefore, rightly termed as artificial variables as opposed to other real decision variables in the problem. Therefore, we must get rid of these variables and must not allow them to appear in the final solution. To achieve this, these variables are assigned a very large per unit penalty in the objective function. This penalty is designated by $-M$ for maximization problems and $+M$ for minimization problems, where M > 0. Value of M is much higher than the cost coefficients of other variables and for hand calculations it is not necessary to assign any specific value to it.
- **Step 3.** Solve the modified linear programming problem by the simplex method.

- While making iterations, using the simplex method, one of the following three cases may arise:
    1. If no artificial variable remains in the basis and the optimality condition is satisfied, then the solution is an optimal feasible solution to the given problem. Also, the original constraints are consistent and none of them is redundant.
    2. If at least one artificial variable appears in the basis at zero level (with zero value in 6-column) and the optimality condition is satisfied, then the solution is optimal feasible (though degenerate) solution to the given problem. The constraints are consistent though redundancy may exist in them. By redundancy is meant that the problem has more than the required number of constraints.
    3. If at least one artificial variable appears in the basis at a non-zero level (with positive value in 6-column) and the optimality condition is satisfied, then the original problem has no feasible solution; for if a feasible solution existed, the artificial variables could be driven to zero, yielding an improved value of the objective function. The problem has no feasible solution either because the constraints are inconsistent or because there are solutions, but none is feasible. In economic terms this means that the resources of the system are not sufficient to meet the expected demands. The final solution to the problem is not optimal since the objective function contains an unknown quantity M. Such a solution satisfies the constraints but does not optimize the objective function and is also called pseudo-optimal solution.

2.12.2 The Two-Phase Method

- In the preceding section we observed that it was frequently necessary to add artificial variables to the constraints to obtain an initial basic feasible solution to an L.P. problem. If the problem is to be solved, the artificial variables must be driven to zero. The two-phase method is another method to handle these artificial variables. Here the L.P. problem is solved in two phases.

Phase I

- In this phase we find an i.b.fs. to the original problem. For this all artificial variables are to be driven to zero. To do this an artificial objective function (w) is created which is the sum of all the artificial variables. This new objective function is then minimized, subject to the constraints of the given (original) problem, using the simplex method. At the end of phase I, three cases arise:

1. If the minimum value of w > 0, and at least one artificial variable appears in the basis at a positive level, then the given problem has no feasible solution and the procedure terminates.
2. If the minimum value of w = 0, and no artificial variable appears in the basis, then a basic feasible solution to the given problem is obtained. The artificial variable column (5) is/are deleted for phase II computations.
3. If the minimum value of w = 0 and one or more artificial variables appear in the basis at zero level, then a feasible solution to the original problem is obtained. However, we must take care of this artificial variable and see that it never becomes positive during phase II computations. Zero cost coefficient is assigned to this artificial variable and it is retained in the initial table of phase II. If this variable remains in the basis at zero level in all phase II computations, there is no problem. However, the problem arises if it becomes positive in some iteration. In such a case, a slightly different approach is adopted in selecting the outgoing variable. The lowest non-negative replacement ratio criterion is not adopted to find the outgoing variable. Artificial variable (or one of the artificial variables if there are more than one) is selected as the outgoing variable. The simplex method can then be applied as usual to obtain the optimal basic feasible solution to the given L.P. problem.

Phase II

- When phase I results in (2) or (3), we go on to phase II to find optimum solution to the given L.P. problem. The basic feasible solution found at the end of phase I is now used as a starting solution for the original problem. In other words, the final table of phase I becomes the starting table of phase II in which the artificial (auxiliary) objective function is replaced by the original objective function. The simplex method is then applied to arrive at the optimum solution. Artificial variables which do not appear in the basis may be deleted.

2.13 Sensitivity Analysis

- Once the optimal solution to a linear programming problem has been attained, two situations may arise which require additional computations
  a. During the formulation it is assumed that the parameters such as market demand, equipment capacity, resource consumption, resource availability, the relevant costs or profits are all known with certainty and do not change over time. In actual practice the markets fluctuate, material and labour costs go up or down, production times change and equipment availability varies from time to time. It is, therefore, desirable to study how the current optimal solution changes when the parameters of the problem get changed. In these problems this information may be more important than the single result provided by the optimal solution. Such an analysis converts the static linear programming solution into a dynamic tool to study the effect of changing conditions such as in business and industry.
  b. The second situation is rather unpleasant, yet one may be encountered with it quite often. After attaining the optimal solution, one may discover that a wrong value of a cost coefficient was used or a particular variable or constraint was omitted or one or more of right-hand constants used were wrong.
- The changes in parameters of the problem may be discrete or continuous. The study of the effect of discrete changes in parameters on the optimal solution is called the sensitivity analysis or the post optimality analysis, while that of continuous changes in parameters is called parametric programming. One way to determine the effects of parameter changes is to solve the problem anew, which may be computationally inefficient. Alternatively, the current optimal solution may be investigated, making use of the properties of the simplex criterion. The second method reduces additional computations considerably and hence forms the subject of the present discussion.

The changes in the parameters of a linear programming problem include:

1. Changes in the right-hand side of the constraints or availability of resources (b_i).
2. Changes in the cost / profit coefficients or cost/profit contribution per unit of decision variables (c_j).
3. Addition of new variables.
4. Changes in the coefficients of constraints or consumption of resources per unit of decision variables (a_{ij})
5. Addition of new constraints.
6. Deletion of variables.
7. Deletion of constraints.

Generally, these parameter changes result in one of the following three cases:

1. The optimal solution remains unchanged i.e., the basic variables and their values remain unchanged.
2. The basic variables remain unchanged but their values change.
3. The basic variables as well as their values are changed.

- While dealing with these changes, one important objective is to find the maximum extent to which a parameter or a set of parameters can be changed so that the current optimal solution remains optimal. In other words, the objective is to determine how sensitive is the optimal solution to the changes in those parameters. Such an analysis is called sensitivity analysis.

2.14 Duality in Linear Programming

- For every L.P. problem (linear programme) there is a related unique L.P. problem (another linear programme) involving the same data which also describes the original problem (programme).
- The given original programme is called the primal programme (P). This programme can be rewritten by transposing (reversing) the rows and columns of the algebraic statement of the problem. Inverting the programme in this way results in dual programme (D). The variables of the dual programme (problem) are known as dual variables or shadow prices of the various resources. A solution to the dual programme may be found in a manner similar to that used for the primal. The two programmes have very closely related properties so that optimal solution of the dual problem gives complete information about the optimal solution of the primal problem and vice versa.
- Duality is an extremely important and interesting feature of linear programming. The various useful aspects of this property are
    a. If the primal problem contains a large number of rows (constraints) and a smaller number of columns (variables), the computational procedure can be considerably reduced by converting it into dual and then solving it. Hence it offers an advantage in many applications.
    b. It gives additional information as to how the optimal solution changes as a result of the changes in the coefficients and the formulation of the problem. This forms the basis of post optimality or sensitivity analysis.
    c. Duality in linear programming has certain far reaching consequences of economic nature. This can help managers to compare the alternative courses of action and their relative values.
    d. Calculation of the dual checks the accuracy of the primal solution.
    e. Duality in linear programming shows that each linear programme is equivalent to a two-person zero-sum game. This indicates that fairly close relationships exist between linear programming and the theory of games.
    f. Duality is not restricted to linear programming problems only but finds application in economics, physics and other fields. In economics it is used in the formulation of input and output systems. In physics it is used in the series circuit and parallel circuit theory.
    g. Economic interpretation of the dual helps the management in making future decisions.

h. Duality is used to solve L.P. problems (by the dual simplex method) in which the initial solution is infeasible.
i. The solution of the dual problem can be used by the decision-maker for planning or augmenting (increasing) the resources.

2.14.1 Dual Problem when Primal is in Canonical Form

- The general linear programming problem in canonical form as discussed earlier:

  maximize

  $$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

  subject to

  $$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1,$$
  $$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2,$$
  $$\vdots \qquad\qquad \vdots$$
  $$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m,$$

  where $x_1, x_2, \ldots x_n \ge 0.$
- If the above problem is referred to as primal, then its associated dual will be

  minimize $W = b_1y_1 + b_2y_2 + \cdots + b_my_m$

  subject to

  $$a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \ge c_1,$$
  $$a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \ge c_2,$$
  $$\vdots \qquad\qquad \vdots$$
  $$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \le c_m,$$

  where $y_1, y_2, \ldots y_n \ge 0.$
- **Rules to construct the dual**
  1) If the primal contains n variables and m constraints, the dual will contain m variables and n constraints.
  2) The maximization problem in the primal becomes the minimization problem in the dual and vice versa.
  3) The maximization problem has ($\le$) constraints while the minimization problem has ($\ge$) cosntraints.
  4) Constraints of $\le$ type in the primal become $\ge$ type in the dual and vice versa.
  5) The coefficient matrix of the constraints of the dual is the transpose of the primal.
  6) A new set of variables appear in the dual.
  7) The constants $c_1, c_2, \ldots, c_n$ in the objective function of the primal appear in the right- hand side of the constraints of the dual.
  8) The constants $b_1, b_2, \ldots, b_m$ in the right-hand side of the constraints of the primal appear in the objective function of the dual.
  9) The variables in both problems are non-negative