3

Transportation and Assignment

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3.1 The Transportation Model

The transportation model is the special case of linear programming problem which deals with the transportation of a product available at several sources to a number of different destinations. The *transportation model* is an important example of a network optimization problem. It can be used for a wide variety of situations such as scheduling, production, investment, plant location, inventory control, personal assignment, and many others, so that the model is not really confined to transportation or distribution only.

3.2 Assumptions in The Transportation Problem

1. The total quantity of items available at different sources is equal to the total requirement at different destinations.
2. Item can be transported conveniently from all sources to destinations.
3. The unit transportation cost of the item from all sources to destinations is certainly and precisely known.
4. The transportation cost on a given route is directly proportional to the number of units shipped on that route.
5. The objective is to minimize the total transportation cost for the organization as a whole and not for individual supply and distribution centres.

3.3 Mathematical/L.P. Model of The Transportation Problem

Transportation model deals with problems concerning as to what happens to te effectiveness function when we associate each of a number of origins (sources) with each of a possibly different number of destinations (jobs).

- Suppose that there are m sources and n destinations.
- Let a_i be the number of supply units available at source $i(i = 1,2,3, \dots, m)$ and let b_j be the number of demand units required at destination $j(j = 1,2,3, \dots, n))$.
- Let c_{ij} represent the unit transportation cost for transporting the units from source i to destination j.
- The objective is to determine the number of units to be transported from source i to destination j so that the total transportation cost is minimum. In addition, the supply limits at the sources and the demand requirements at the destinations must be satisfied exactly.
- The above information can be put in the form of a general matrix as shown below:

		Destination				
		1	2	…	n	Supply
Source	1	$c_{11} \cdot x_{11}$	$c_{12} \cdot x_{12}$	…	$c_{1n} \cdot x_{1n}$	a_1
	2	$c_{21} \cdot x_{21}$	$c_{22} \cdot x_{22}$	…	$c_{2n} \cdot x_{2n}$	a_2
	⋮	⋮	⋮	…	⋮	⋮
	m	$c_{m1} \cdot x_{m1}$	$c_{m2} \cdot x_{m2}$	…	$c_{mn} \cdot x_{mn}$	a_m
	Demand (Requirement)	b_1	b_2	…	b_n	

If x_{ij} ($x_{ij} \geq 0$) is the number of units shipped from source i to destination j, then the equivalent linear programming model will be given as follows:

Find $x_{ij}(i = 1, 2, 3, \dots, m; j = 1, 2, 3, \dots, n)$ in order to

$$Minimize\ (total\ cost)\ Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$Subject\ to\ the\ constraints$

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1,2,3, \dots \dots .., m,$$

and

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1,2,3, \dots \dots .., n,$$

where

$$x_{ij} \geq 0$$

The two sets of constraints will be consistent i.e., the system will be in balance if

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Note that the transportation problem will have a feasible solution if the above restriction is satisfied. Thus $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ is necessary as well as sufficient condition for a transportation problem to have a feasible solution.

Problems that satisfy this condition are called balanced transportation problems. Techniques have been developed for solving *balanced or standard* transportation problems only. It follows that any non-standard problem in which the supplies and demands do not balance, must be converted to balanced transportation problem before it can be solved. This can be done by the use of a dummy source/destination.

3.4 The Solution of the Transportation Model

Following steps are involved in the solution of transportation model.

1. Balance the problem if it is not.
2. Obtain the initial feasible solution by
    a. North-West Corner Method (NWCM) or,
    b. Inspection Method / Least Cost Method (LCM) or,
    c. Penalty Method / Vogel's Approximation Method (VAM).
3. Test the optimality of the solution either by
    a. Stepping Stone Method or,
    b. Modified Distribution Method (MODI).

3.4.1 Methods for Finding Initial Solution

3.4.1.1 North-West Corner Method (NWCM)

It is the simplest procedure used to generate an initial feasible solution. It is so called because we begin with the north-west or upper-left corner cell of the transportation table. Various steps of this method can be summarized as follows:

1. Select the north-west (upper-left) corner cell of the transportation table and allocate as many units as possible (i.e. equal to the minimum between available supply and demand.)
2. Adjust the supply and demand numbers in the respective rows and columns allocation.
3. If the supply for the first row is exhausted, then move down to the first cell in the second row and first column and go to step 2. **OR**
   If the demand for the first column is satisfied, then move horizontally to the next cell in the second column and first row and got to step 2.
4. If for any cell, supply equals demand, then the next allocation can be made in cell either in the next row or column.
5. Continue the procedure until the total available quantity is fully allocated to the cells as required.

3.4.1.2 Least Cost Method (LCM)

Various steps of this method can be summarized as under:

1. Select the cell with the lowest transportation cost among all the rows or columns of the transportation table.
   If the minimum cost is not unique, then select arbitrarily any cell with this minimum cost. For a better solution select the cell where the highest units can be allotted.
2. Allocate as many units as possible to the cell determined in step 1 and eliminate that row (or column) in which either supply is exhausted or demand is satisfied.
3. Repeat steps 1 and 2 for the reduced table until the entire supply at different factories is exhausted to satisfy the demand at different warehouses.

3.4.1.3 Vogel's Approximation Method (VAM) / Penalty Method

This method is preferred over the other two methods described above because the initial feasible solution obtained is either optimum or very much nearer to an optimum solution.

1. Calculate penalties for each row and column by taking the difference between the smallest and the next smallest unit transportation cost in the same row and column respectively.
2. Identify the row or column with the largest penalty and allocate the maximum possible quantity to the lowest cost cell in that selected row or column so as to exhaust either supply or demand. If there is a tie in the penalty or minimum cost select that row or column which will have maximum possible assignments.
3. Reduce the row supply or column demand by the amount assigned to the cell.
4. If the row supply is now zero, eliminate the row and if the column demand is zero, eliminate the column.
5. Recompute the row and column differences for the reduced transportation table, omitting rows and columns crossed out in the preceding step.
6. Repeat the above procedure until the entire supply at factories is exhausted to satisfy demand at different warehouses.

3.4.2 Methods for Finding an Optimal Solution

Once an initial solution has been found, the next step is to check its optimality.

3.4.2.1 Stepping Stone Method

In this method, we calculate the net cost change that can be obtained by introducing any of the unoccupied cells into the solution.

The stepping stone method can be summarized as under:

1. Determine an initial basic feasible solution using any of the three methods.
2. Make sure that the no. of occupied cells is exactly equal to m+n-1, where m is no. of rows and n is no. of columns.
3. Evaluate the cost effectiveness goods via transportation roots not currently in solution by the following five steps:
    a. Select an unoccupied cell, where a shipment should be made.
    b. Starting from this cell, formed a closed path (or loop) through at least three occupied cells and eventually returning to the same occupied cell. In the loop formulation only right angle turn is allowed and, therefore, skips all other cells which are not at the turning points. (Right angle turns in this path are permitted only at occupied cells)
    c. Assigning (+) and (-) alternatively on each corner cell of the closed path just traced, starting with a (+) sign at the unoccupied cell to be evaluated.
    d. Compute the net change in the cost along the closed path by adding together the unit transportation cost containing (+) and then subtracting unit cost containing (-).
4. Check the sign of each of the net changes. If all the net changes computed are greater than or equal to zero, an optimal solution has been reached. If not, it is possible to improve the current solution.
5. Select the unoccupied cell having the highest negative net cost change and determine the maximum no. of units that can be assigned to a cell marked with a minus sign on the closed path corresponding to this cell. Add this number to the unoccupied cell and to all other cells on the path marked with a plus sign. Subtract this number from cells on the closed path marked with a minus sign.
6. Go to step 2 and repeat the procedure until we get an optimal solution.

3.4.2.2 Modified Distribution Method (MODI Method)

The modified distribution method has a pattern similar to the stepping stone method.

1. Determine an initial basic feasible solution considering of (m+n-1) allocations in independent positions using any of the three methods.
2. Determine a set of numbers for each row and each column. To compute u_i (i = 1, 2, 3, …., m) for each row and v_j (j = 1, 2, 3, ….., n) for each column, set $c_{ij} = u_i + v_j$ for each of the m+n-1 occupied cells.
3. Compute the opportunity cost by $\Delta_{ij} = c_{ij} - (u_i + v_j)$ for each of the unoccupied cells.
4. Check the sign of opportunity cost. If one or more unoccupied cell has a negative opportunity cost, the given solution is not optimum.
5. Select the unoccupied cell with the largest negative opportunity cost as the cell to be included in the next solution.

6. Draw a closed path for unoccupied cell selected in step 5.
7. Assign alternate + and – signs on the corner point of the closed path.
8. Determine the maximum number of units that should be shipped to this unoccupied cell.
9. Repeat the whole procedure until an optimal solution is attained.
10. Finally, obtain the total transportation cost for the new solution.

3.5 Special Variations in The Transportation Model

3.5.1 Prohibited and Preferred Routes

In many practical situations, some of the routes in transportation are prohibited for some external reasons such as unexpected floods, weight or size conditions, transportation strikes and road construction, etc.

Such restrictions (or prohibitions) can be handled in the transportation problem by assigning a very high cost (say M or ∞) to the prohibited routes to ensure that these routes will not be included in the optimal solution and then the problem is solved in the usual manner.

3.5.2 Maximization Transportation Problem

When origins are related to destinations by a profit function instead of cost, the objective is to maximize instead of minimize.

The most convenient way to deal with the maximization case is to transform the profits to relative costs. The transformation is conducted by subtracting the profit associated with each cell from the largest profit in the matrix. In this way, the largest profit shows a zero relative cost. The solution procedures for relative costs are identical to those already described.

After an optimal solution to the transformed minimization problem is determined, relative costs in occupied cells are restated in their original profit functions, and the total profit is calculated according to quantities in the optimal distribution pattern.

3.6 Degeneracy in the transportation problem

An initial feasible solution to m origins and n destinations transportation problem consists of $(m + n - 1)$ basic variables, i.e., the number of occupied cells is less than $(m + n - 1)$. But if the number of occupied cells is less than $(m + n - 1)$ at any stage of the solution, then the problem is said to have a degenerate solution.

Degeneracy can occur at two stages: either (i) at the initial solution, or (ii) during testing of the optimal solution.

Case 1: Degeneracy occurs at the initial solution:

To resolve degeneracy at the initial stage, we make use of an artificial quantity, denoted by the Greek letter ε $(epsilon)$ in one or more of the unoccupied cells so that the number of occupied cells is $(m + n - 1)$. The quantity ε is so small that it does not affect the supply & demand constraints. The ε value of which is assumed to be zero when actually used in the movements of goods from one cell to another is generally placed in the unoccupied cell with the lowest transportation cost and that cell is considered to be occupied.

Case 2: Degeneracy occurs during the testing of the optimal solution:

Degeneracy during the solution stage occurs, when the inclusion of the unoccupied cells with maximum negative opportunity cost results in vacating of two or more occupied cells simultaneously. This type of degeneracy is resolved by allocating ε to one or more of the recently vacated cells to complete the required $(m + n - 1)$ number of occupied cells. The problem is then solved in the usual manner.

3.7 The Assignment Model

The assignment model refers to another special class of linear programming problem.

An assignment problem concerns as to what happens to the effectiveness function when we associate each of a number of ***origins*** with each of the same number of ***destinations***. Each resource or facility is to be associated with one and only one job (destination) and associations are to be made in such a way so as to minimize (or maximize) the total effectiveness. Resources are not divisible among jobs, nor are jobs divisible among resources.

3.8 Mathematical/L.P. Model of The Assignment Problem

Given n facilities (or resources) and n jobs and the given effectiveness (in terms of cost, time, profit, etc.) of each facility for each job, the problem lies is assigning each resource to one and only one activity so that the given measure of the effectiveness is optimized.

		Jobs				
		1	2	...	n	Supply (a_i)
Facilities	1	c_{11}	c_{12}	...	c_{1n}	1
	2	c_{21}	c_{22}	...	c_{2n}	1
	⋮	⋮	⋮	...	⋮	⋮
	m	c_{n1}	c_{n2}	...	c_{nn}	1
	Demand (b_j)	1	1	...	1	

The above matrix represents the assignment of n facilities to n jobs. c_{ij} is the cost of assigning ith facility to jth job and x_{ij} represents the assignment of ith facility to jth job. If ith facility can be assigned to jth job, $x_{ij} = 1$, otherwise zero. The objective is to make assignments that minimize the total assignment cost or maximize the total associated gain.

Thus an assignment problem can be represented by $n \times n$ matrix which constitutes $n!$ possible ways of making assignments.

Mathematically the assignment model can be represented as follows:

Let x_{ij} denote the assignment of a facility i to job j such that

$x_{ij} = 0$ if the ith facility is not assigned to jth job

$= 1$ if the ith facility is assigned to jth job

Then the model is given by

$$minimize\ Z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

Subject to constraints

$$\sum_{j=1}^{n} x_{ij} = 1, \qquad for\ i = 1,2,3, \dots \dots, n\ (one\ job\ is\ assigned\ to\ the\ i^{th}\ facility)$$

$$\sum_{i=1}^{n} x_{ij} = 1, \qquad for\ j = 1,2,3, \dots \dots, n\ (one\ facility\ is\ assigned\ to\ the\ j^{th}\ job)$$

and

$$x_{ij} = 0\ or\ 1, for\ all\ i\ and\ j$$

It may be noted that if the last condition is replaced by $x_{ij} \geq 0$, we have a transportation model with all requirements and available resources equal to 1.

3.9 Comparison between The Assignment and The Transportation model

The assignment model may be regarded as a special case of the transportation model. However, the *transportation algorithm* is not very useful to solve this model because of degeneracy.

Similarities

1. Both are special types of linear programming problems.
2. Both have an objective function, structural constraints, and non-negativity constraints. And the relationship between variables and constraints are linear.
3. The coefficients of variables in the solution will be either 1 or zero in both cases.
4. Both are basically minimization problems. For converting them into maximization problem same procedure is used.

Differences

*Table **3.1** - Comparison between transportation and assignment Model*

The Transportation Model	The Assignment Model
The problem may have rectangular matrix or square matrix.	The matrix of the problem must be a square matrix.
The rows and columns may have any number of allocations depending on the rim conditions.	The rows and columns must have one to one allocation. Because of this property, the matrix must be a square matrix.
The basic feasible solution is obtained by the northwest corner method or LCM method or VAM	The basic feasible solution is obtained by the Hungarian method or Flood's technique or by Assignment algorithm.
The optimality test is given by the stepping stone method or by the MODI method.	Optimality test is given by drawing minimum number of horizontal and vertical lines to cover all the zeros in the matrix.
The rim requirement may have any positive numbers.	The rim requirements are always 1 each for every row and one each for every column.
The transportation algorithm can be used to solve the assignment model.	The assignment algorithm can not be used to solve the transportation model.

3.10 The Hungarian Method (Minimization in Assignment)

The Hungarian method of assignment is an efficient method of finding the optimal solution.

Theorem:

In an assignment problem, if a constant quantity is added or subtracted to every element of the given cost matrix, an assignment that minimizes the total cost in one matrix also minimizes the total cost in the other.

The computational procedure of solving assignment problem (where no. of rows are equal to the no. of columns) can be summarized in the following steps:

1. **Prepare a square matrix:**

   If the n facilities and n jobs are not equal than make it equal by adding the dummy rows or columns to complete the square matrix. If it is given in problem than go for the next step.

2. **Reduce the matrix:**

   a. Row reduction: Subtract the minimum entry of each row from all the entries of the respective row in the cost matrix, after row reduction check if each and every column has at least one zero in it then go to step 3 otherwise do column reduction.

   b. Column reduction: After completion of row reduction, subtract the minimum entry of each column from all the entries of the respective column.

3. **Make assignments in the opportunity cost matrix:**

   a. Starting with the first row of the matrix received in the first step, examine the rows successively one by one until a row containing exactly one zero is found. Then an experimental assignment indicated by '□' is marked to that zero. Now cross (x) all the zeros in the column in which the assignment is made. This procedure should be adopted for each row assignment.

   b. When the set of rows has been completely examined, an identical procedure is applied successively to columns. Starting with column 1, examine all columns until a column containing exactly one zero is found. Then make an assignment in that position and cross (x) other zeros in the row in which the assignment was made.

      Continue these successive operations on rows and columns until all zeros have either been assigned or crossed-out.

   c. If for a row and for column there are two or more zeros and one cannot be chosen for inspection, choose the assigned zero cell arbitrarily.

   d. Repeat the operations of (a) to (c) successively until one of the following situations arises:

4. **Optimality criterion:**

   a. If all the zeros in row/column are either marked with box '□' or cross (x) and there is exactly one assignment in each row and in each column. In such a case optimal assignment policy for the given problem is obtained.

   b. There may be some row or column without assignment, i.e., the total no. of marked zeros is less than the order of the matrix. In such a case, proceed to step 5.

5. **Revise the opportunity cost matrix:**

Draw the minimum no. of vertical and horizontal lines necessary to cover all zeros in the reduced matrix obtained from step 3 by adopting the following procedure:

a. Mark check (✓) to all those rows where no assignment has been made.

b. Examine the checked (✓) rows. If any zero cells occur in these rows, check (✓) the respective columns that contain those zeros.

c. Examine the checked (✓) columns. If any assigned zero cells occur in those columns, check (✓) the respective rows that contain those assigned zeros.

d. The process may be repeated until no more rows or columns can be checked.

e. Draw straight lines through all unchecked rows and through all checked columns.

6. **Develop the new revised opportunity cost matrix:**

   a. Select the smallest element among all the uncovered elements.

   b. Subtract this smallest element to all those elements which are not covered.

   c. Add this smallest element to all those elements which lie at the intersection of two lines.

7. **Go to step 3** and repeat the procedure until the no. of assignments becomes equal to the no. of rows or columns. In such a case, we shall observe that each row/column has an assignment. Thus the current solution is an optimal solution.

3.11 Special Variations in the Assignment Problem

3.11.1 Multiple Optimal Solutions:

Sometimes, it is possible to have two or more ways to cross out all zero elements in the final reduced matrix for a given problem. In such cases, there will be alternate optimal solutions with the same cost. Alternate optimal solutions offer great flexibility to the management since they can select the one which is most suitable to its requirement.

3.11.2 Unbalanced Assignment Problem (Non-square Matrix):

Such a problem is found to exist when the no. of facilities is not equal to the no. of jobs. Since the Hungarian method of solution requires a square matrix, fictitious facilities or jobs may be added and zero costs are to be assigned to the corresponding cells of the matrix. These cells are then treated the same way as the real cost cells during the solution procedure.

3.11.3 Maximization Assignment Problem:

Sometimes the assignment problem may deal with maximization of the objective function. The maximization problem has to be changed to minimization before the Hungarian method may be applied. This transformation may be done in either of the following two ways:

a. By subtracting all the elements from the largest element of the matrix.

b. By multiplying the matrix elements by -1.

The Hungarian method can then be applied to this equivalent minimization problem to obtain the optimal solution.

3.11.4 Restrictions on Assignment:

Sometimes technical, space, legal or other restrictions do not permit the assignment of a particular facility to a particular job. Such problems can be solved by assigning a very heavy cost (infinite cost) to the corresponding cell. Such a job will then be automatically excluded from further consideration (making assignments).

3.11.5 Travelling Salesman Problem

A salesman has to call on n cities. He is now stationed at city 1. Given the associated cost or distance Cij for going from city i to city j, he has to plan his tour so that he completes the tour with minimum total cost or distance, and return, at the end of the tour, to station 1.

The mathematical formulation for the above problem can be as follows:

The mathematical statement of the problem is to decide variables x_{ij} = 1 or 0 for all i and j.

x_{ij} = 1, if the salesman goes from city i to j.

x_{ij} = 0, if the salesman does not go from city i to j.

With n tasks and n agents let c_{ij} be the cost for ith agent performing jth task, then

$$minimize\ Z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

Subject to constraints

$$\sum_{j=1}^{n} x_{ij} = 1,$$

$$\sum_{i=1}^{n} x_{ij} = 1,$$

and

$$x_{ij} \geq 0, for\ all\ i\ and\ j$$

with the additional constraints that no city is visited twice before the tour of all the cities is completed and that going from city i directly to i is not permitted, which means $c_{ii} = \infty$.

For solving travelling salesman problem for city 1,2,3…n. Assign the highest cost at the cell 1-1, 2-2, 3-3, 4-4, ……, n-n. So that this assignment will be excluded from the solution and then follow the normal procedure of assignment.