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Game Theory

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7.1 Introduction

Life is full of conflict and competition. Numerous examples involving adversaries in conflict include parlor games, military battles, political campaigns, advertising and marketing campaigns by competing business firms, and so forth. A basic feature in many of these situations is that the final outcome depends primarily upon the combination of strategies selected by the adversaries.

The theory of games is a mathematical theory that deals with the general features of competitive situations. This theory is helpful when two or more individuals or organizations with conflicting objectives try to make decisions.

The theory of games is based on the minimax principle put forward by J. Von Neumann which implies that each competitor will act so as to minimize his maximum loss (or maximize his minimum gain) or achieve the best of the worst.

7.2 Characteristics of Games

Following are the characteristics of the Games Theory problem.

- A game has a finite number of players.
- Each player has a finite number of strategies & these strategies are known to all concerned players.
- Choice of a particular strategy is exercised simultaneously by all players.
- The game results in a finite outcome which is the net consequence of actions of all players. The outcome is in terms of gain.

7.3 Terminologies used in Games Theory

1) **Game**
   It is an activity, between two or more persons, involving actions by each one of them according to a set of rules, which results in some gain (+ve, -ve or zero) for each.
2) **Player**
   Each Participant playing a game is called a player.
3) **Play**
   A play of the game is said to occur when each player chooses one of his courses of action.
4) **Strategy**
   It is the predetermined rule by which a player decides his course of action from his list of courses of actions during the game.
   a. **Pure Strategy**
      A pure strategy is one in which a player chooses only one course of action at every play.
   b. **Mixed Strategy**
      A mixed strategy is one in which a player chooses different strategies in certain frequency of occurrence i.e. in fixed proportion from time to time, when the game is repeated.
   c. **Optimal Strategy**
      The strategy that puts the player in the most preferred position irrespective of the strategy of his opponents is called an optimal strategy. Any deviation from this strategy would reduce his payoff.
5) **Saddle point**
   The game value is called the saddle point in which each player has a pure strategy. If the saddle point exists, the game is said to be stable.

6) **Zero-sum game**
A zero-sum game is one in which the sum of gains to all players at the end of each play is zero.

7) **Pay-off Matrix**
It is the outcome of the game. The Pay-off matrix is the table showing the amounts received by the player named at the left-hand side after all possible plays of the game. The payment is made by a player named at the top of the table.

Strategies

		Player B		
		1	2	3
Player A	1	a_{11}	a_{12}	a_{13}
	2	a_{21}	a_{22}	a_{23}
	3	a_{31}	a_{32}	a_{33}

← Pay-off Matrix

8) Non-zero sum game
It is the game in which the sum of payments to all the players, after the play of the game is not zero. Any player may receive or make some payments.

7.4 Two-Person Zero-Sum Game

Basic assumptions of the game:

1) Each player has available to him a finite number of possible courses of action. The list may not be the same for each player.
2) Player A attempts to maximize gains and player B minimize losses.
3) The decisions of both players are made individually prior to the play with no communication between them.

7.4.1 Games with Saddle Point

Minimax and Maximin Principle

- Consider the pay matrix of a game that represents the pay-off of player A. Now, the objective of the study is to know how these players must elect their respective strategies so that they may optimize their pay-off. Such a decision-making criterion is referred to as the minimax-maximin principle.
- For player A minimum value in each row represents the least gain (pay-off) to him if he chooses his particular strategy. These are written in the matrix by row minima. He will then select the strategy that gives the largest gain among the row minimum values. This choice of player A is called the maximin principle, and the corresponding gain is called the maximin value of the game denoted by v.
- For player B (who is assumed to be the loser), the maximum value in each column represents the maximum loss to him if he chooses his particular strategy. These are written in the pay-off matrix by column minima. He will then select the strategy that gives minimum loss among the column maximum values. This choice of player B is called the minimax principle, and the corresponding loss is the minimax value of the game denoted by v.
- Saddle point: A saddle point of a pay-off matrix is that position in the pay-off matrix where maximum of row minima coincides with the minimum of the column maxima. The saddle point need not be unique.
- Value of the game: The pay-off of the saddle point is called the value of the game denoted by v.

- Fair game: A game is said to be fair if v = 0 = v.

7.4.2 Games without Saddle Point

7.4.2.1 *Mixed Strategies*

- There are some games for which no saddle point exists. In such cases both the players must determine an optimal combination of strategies to find a saddle (equilibrium) point. The optimal strategy combination for each player may be determined by assigning to each strategy its probability of being chosen.
- The strategies so determined are called mixed strategies because they are probabilistic combination of available choices of strategy.
- The value of game obtained by the use of mixed strategies represents least pay-off which player A can expect to win and the least which player B can lose.
- A mixed strategy game can be solved by different solution methods such as
  - a. Algebraic method
  - b. Analytical or calculus method
  - c. Matrix method
  - d. Graphical method, and
  - e. Linear programming method.

7.4.2.2 *Dominance Property of Reducing the Size of the Game*

- We can sometimes reduce the size of a game's pay-off matrix by eliminating a course of action that is so inferior to another as never to be used. Such a course of action is said to be dominated by the other. The concept of dominance is especially useful for the evaluation of two-person zero-sum games where a saddle point does not exist.
- General rule
  1) If all the elements of a row, say k^{th}, are less than or equal to the corresponding elements of any other row, say r^{th}, then k^{th} row is dominated by the r^{th} row.
  2) If all the elements of a column, say k^{th} are greater than or equal to the corresponding elements of any other column, say r^{th}, then k^{th} column is dominated by r^{th} column.
  3) Omit dominated rows or columns.
  4) If some linear combination of some rows dominates i^{th} row, then i^{th} row will be deleted. The similar argument follows for columns.

7.5 Solution Methods of Games Without Saddle Point

7.5.1 Arithmetic Method or Method of Oddments (2x2 games)

- In these cases with each move, some probability is attached. Let the pay-off matrix is given by

		Player B	
		I	II
Player A	I	a_{11}	a_{12}
	II	a_{21}	a_{22}

- Find out the oddments as follows

  $\Delta_1 = |a_{21} - a_{22}|$ & $\Delta_2 = |a_{11} - a_{12}|$

$\Delta_3 = |a_{12} - a_{22}|$ & $\Delta_2 = |a_{11} - a_{21}|$

- From the oddments find the frequency of the strategies used by the players as follows:

$$x_1 = \frac{\Delta_1}{\Delta_1 + \Delta_2} \text{ \& } x_2 = \frac{\Delta_2}{\Delta_1 + \Delta_2}$$

$$y_1 = \frac{\Delta_3}{\Delta_3 + \Delta_4} \text{ \& } y_2 = \frac{\Delta_4}{\Delta_3 + \Delta_4}$$

- The value of the game can be found using any of the following four equations:

$Va_1 = a_{11}y_1 + a_{12}y_2$
$Va_2 = a_{11}y_1 + a_{12}y_2$
$Vb_1 = a_{11}x_1 + a_{21}x_2$
$Vb_2 = a_{12}x_1 + a_{22}x_2$

7.5.2 Algebraic Method (2 x 2 Games)

- In the algebraic method, it is presumed that x_1 represents the fraction of time or frequency when player A uses the first strategy and x_2 or $(1-x_1)$ represents the time when player A uses the alternate strategy. Likewise, y_1 represents the fraction of time or frequency when player B uses the first strategy and y_2 or $(1-y_1)$ the time when player B uses the alternate strategy.

		Player B	
		I	II
Player A	I	a_{11}	a_{12}
	II	a_{21}	a_{22}

- Here

$$V_{a_1} = V_{a_2}$$

$$\therefore a_{11}y_1 + a_{12}y_2 = a_{21}y_1 + a_{22}y_2 \qquad \textit{Eq. (7.1)}$$

and

$$y_1 + y_2 = 1 \qquad \textit{Eq. (7.2)}$$

- Similarly

$$V_{b_1} = V_{b_2}$$

$$\therefore a_{11}x_1 + a_{21}x_2 = a_{12}x_1 + a_{22}x_2 \qquad \textit{Eq. (7.3)}$$

and

$$x_1 + x_2 = 1 \qquad \textit{Eq. (7.4)}$$

- *Eq. (7.1)* and *Eq. (7.2)* are two independent equations connecting y_1 and y_2. Hence, values of y_1 and y_2 can be decided. Likewise values of x_1 and x_2 can be decided from equations *Eq. (7.3)* and *Eq. (7.4)*.
- With the values of x_1, x_2, y_1 and y_2 known value of the game is given by any of the equation from *Eq. (7.1)* or *Eq. (7.3)*.

7.5.3 Subgame Method to Solve (2 × n Or m × 2) Mixed Strategy Games

- Most of the 2 × n or m × 2 games can be reduced by the method of dominance to a 2 × 2 game but there are cases when the reduction becomes impossible or partially possible. This results in a game which is still larger than the 2 × 2 game. To solve such games, one of the methods is "solution by sub games".
- Steps involved in the basic procedure are as under:
  a. **Check for saddle point**: If saddle point does not exist, the game is a mixed strategy game and we go to the step b.
  b. **Check for dominance:** Check for dominance in order to reduce the size of the pay-off matrix to 2 × 2 game. If there is no dominance move to step III.

c. Break the matrix into a number of smaller 2 × 2 games and solve each game independently.
d. Select a subgame which optimizes the gains of the player playing more than two strategies.
e. Verify the selected sub-game for optimality.

7.5.4 Graphical Method to Solve a (2 × n) Or (m × 2) Games

- The subgame method is feasible for small values of n, because the larger value of n will yield a large number of 2 × 2 subgames. In such cases graphical method can be used to solve (m × 2) or (2 × n) games. It indicates which of the 2 × 2 subgames is optimal one. In addition to this the graphical method indicates the optimal game value. This is comparatively a faster method to solve (2 × n) games.