2

Graphical and Analytical Linkage Synthesis

Contents

2.1 Synthesis of Mechanisms

The ***synthesis of the mechanism*** is the design or creation of a mechanism to produce the desired output motion for a given input motion. In other words, the synthesis of mechanism deals with the determination of proportions of a mechanism for the given input and output motion.

In the application of synthesis, to the design of a mechanism, the problem divides itself into the following three parts:

- **Type synthesis**: Type Synthesis refers to the kind of mechanism selected; it might be a linkage, a geared system, belts, and pulleys, or even a cam system.

  This beginning phase of the total design problem usually involves design factors such as manufacturing processes, materials, safety space and economics. The study of kinematics is usually only slightly involved in type synthesis.

- **Number synthesis**: Number synthesis deals with the number of links, and the number of joints or pairs that are required to obtain certain mobility. Number synthesis is the second step in design following type synthesis.

- **Dimensional synthesis**: The proportions or lengths of the links necessary to satisfy the required motion characteristics.

In designing a mechanism, one factor that must be kept in mind is that of the accuracy required of the mechanism. Sometimes, it is possible to design a mechanism that will theoretically generate a given motion. The difference between the desired motion and the actual motion produced is known as ***structural error***.

In addition to this, there are errors due to manufacturing. The error resulting from tolerances in the length of links and bearing clearances is known as ***mechanical error***.

2.1.1 Classifications of Synthesis Problem

a) Function Generation

A frequent requirement in design is that of causing an output member to rotate, oscillate or reciprocate according to a specified function of time or function of input motion. This is called function generation.

A simple example is that of synthesizing a four-bar linkage to generate the function y=f(x). In this case, x would represent the motion (crank angle) of input crank, and the linkage would be designed so that the motion (angle) of the output rocker would approximate the function y.

Other examples of function generation are as follows:

1. In a conveyor line the output member of a mechanism must move at the constant velocity of the conveyor while performing some operations – Ex. bottle capping, return, pick up the next cap and repeat the operation.
2. The output member must pause or stop during its motion cycle to provide time for another event. The second event might be a sealing, stapling, or fastening operation of some kind.
3. The output member must rotate at a specified non-uniform velocity function because it is geared to another mechanism that requires such a rotating motion.

b) Path Generation:

The second type of synthesis problem is called path generation. This refers to a problem in which a coupler point is to generate a path having a prescribed shape. Common requirements are that a portion of the path is a circular arc, elliptical or straight line. Sometimes it is required that the path cross over itself as in a figure-of-eight.

c) Body Guidance:

The third general class of synthesis problem is called body guidance. Here we are interested in moving an object from one position to another.

The problem may call for a simple translation or a combination of translation and rotation (JCB example). In the construction industry, for example, heavy parts such as scoops and bulldozer blades must be moved through a series of prescribed positions.

2.2 Freudenstein's Equation (Synthesis a four-bar mechanism)

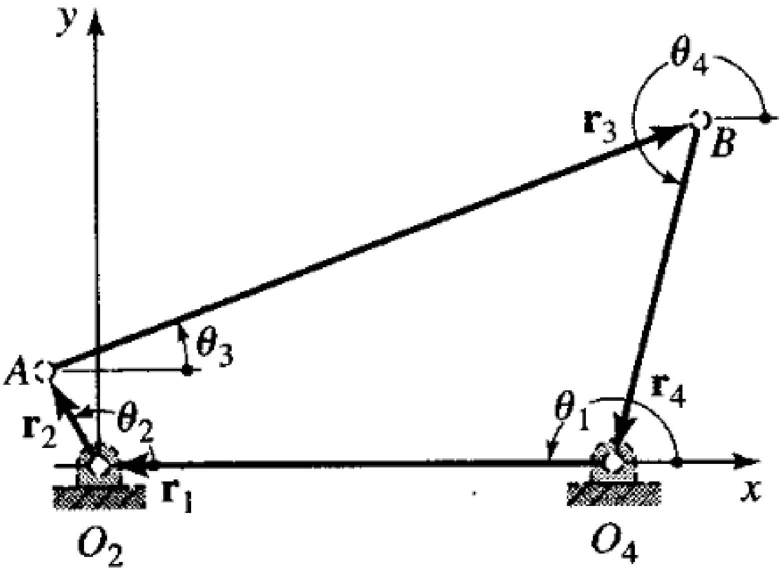

Fig.2.1 – Four bar mechanism

- Replace the link of four-bar linkage by position vector and write the vector equation.

$$r_1 + r_2 + r_3 + r_4 = 0$$

In complex polar notation above equation can be written as

$$r_1 e^{j\theta_1} + r_2 e^{j\theta_2} + r_3 e^{j\theta_3} + r_4 e^{j\theta_4} = 0$$

Above equation is transformed into complex rectangular form by putting

$$\boldsymbol{e^{j\theta} = \cos\theta + j \cdot \sin\theta}.$$

$$\therefore r_1(\cos\theta_1 + j\cdot\sin\theta_1) + r_2(\cos\theta_2 + j\cdot\sin\theta_2) + r_3(\cos\theta_3 + j\cdot\sin\theta_3) + r_4(\cos\theta_4 + j\cdot\sin\theta_4) = 0$$

- Now, if the real and imaginary components of the above equation are separated, we obtain the two algebraic equations

$$r_1\cos\theta_1 + r_2\cos\theta_2 + r_3\cos\theta_3 + r_4\cos\theta_4 = 0$$

$$r_1\sin\theta_1 + r_2\sin\theta_2 + r_3\sin\theta_3 + r_4\sin\theta_4 = 0$$

In the above equation $\boldsymbol{\sin\theta_1 = 0}$ and $\boldsymbol{\cos\theta_1 = -1}$

$$\therefore -r_1 + r_2\cos\theta_2 + r_3\cos\theta_3 + r_4\cos\theta_4 = 0$$

$$r_2\sin\theta_2 + r_3\sin\theta_3 + r_4\sin\theta_4 = 0$$

- Now,

$$\therefore r_3\cos\theta_3 = r_1 - r_2\cos\theta_2 - r_4\cos\theta_4$$

$$\therefore r_3\sin\theta_3 = -r_2\sin\theta_2 - r_4\sin\theta_4$$

- Squaring and Adding both the equations

$$r_3^2(\cos^2\theta_3 + \sin^2\theta_3) = (r_1 - r_2\cos\theta_2 - r_4\cos\theta_4)^2 + (-r_2\sin\theta_2 - r_4\sin\theta_4)^2$$

$$\therefore r_3^2 = (r_1 - a)^2 + (-r_2\sin\theta_2 - r_4\sin\theta_4)^2$$

$$= r_1^2 - 2ar_1 + a^2 + r_2^2\sin^2\theta_2 + 2r_2r_4\sin\theta_2\sin\theta_4 + r_4^2\sin^2\theta_4$$

$$= r_1^2 - 2(r_2\cos\theta_2 + r_4\cos\theta_4)r_1 + (r_2\cos\theta_2 + r_4\cos\theta_4)^2 + r_2^2\sin^2\theta_2 + 2r_2r_4\sin\theta_2\sin\theta_4 + r_4^2\sin^2\theta_4$$

$$= r_1^2 - 2r_1r_2\cos\theta_2 - 2r_1r_4\cos\theta_4 + r_2^2\cos^2\theta_2 + 2r_2r_4\cos\theta_2\cos\theta_4 + r_4^2\cos^2\theta_4 + r_2^2\sin^2\theta_2 + 2r_2r_4\sin\theta_2\sin\theta_4 + r_4^2\sin^2\theta_4$$

$$= r_1^2 + r_2^2 + r_4^2 - 2r_1r_2\cos\theta_2 - 2r_1r_4\cos\theta_4 + 2r_2r_4(\cos\theta_2\cos\theta_4 + \sin\theta_2\sin\theta_4)$$

$$\therefore r_3^2 - r_1^2 - r_2^2 - r_4^2 + 2r_1r_2\cos\theta_2 + 2r_1r_4\cos\theta_4 = 2r_2r_4\cos(\theta_2 - \theta_4)$$

- Dividing both the sides by $2r_2r_4$

$$\therefore \frac{r_3^2 - r_1^2 - r_2^2 - r_4^2}{2r_2r_4} + \frac{r_1}{r_4}\cos\theta_2 + \frac{r_1}{r_2}\cos\theta_4 = \cos(\theta_2 - \theta_4)$$

$$K_1\cos\theta_2 + K_2\cos\theta_4 + K_3 = \cos(\theta_2 - \theta_4)$$

Where

$$K_1 = \frac{r_1}{r_4}, \qquad K_2 = \frac{r_1}{r_2}, \qquad K_3 = \frac{r_3^2 - r_1^2 - r_2^2 - r_4^2}{2r_2r_4}$$

- Freudenstein's equation enables us to perform this same task by analytical means. Thus suppose we wish the output lever of a four-bar linkage to occupy the position $\emptyset_1, \emptyset_2, \text{and } \emptyset_3$ corresponding to the angular positions $\psi_1, \psi_2, \text{and } \psi_3$ of the input lever. We simply replace θ_2 with ψ_i, $\theta_4$with $\emptyset_i$, and write the equation three times, once for each position.

$$K_1\cos\psi_1 + K_2\cos\phi_1 + K_3 = \cos(\psi_1 - \phi_1)$$

$$K_1\cos\psi_2 + K_2\cos\phi_2 + K_3 = \cos(\psi_2 - \phi_2)$$

$$K_1\cos\psi_3 + K_2\cos\phi_3 + K_3 = \cos(\psi_3 - \phi_3)$$

2.3 Two-Position Synthesis of Slider-Crank Mechanisms

The centered slider-crank mechanism has a stroke B_1B_2 equal to twice the crank radius r_2 ($B_1B_2 = 2r_2$). As shown, the extreme positions of B_1 and B_2, also called limiting positions of the slider, are found by constructing circular arcs through O_2 of length ($r_3 - r_2$) and ($r_3 + r_2$), respectively.

In general, the centered slider-crank mechanism must have $r_3 > r_2$. However, the special case of $r_1 = r_2$ results in the *isosceles slider-crank mechanism,* in which the slider reciprocates through O_2 and the stroke $4 \times r_2$.

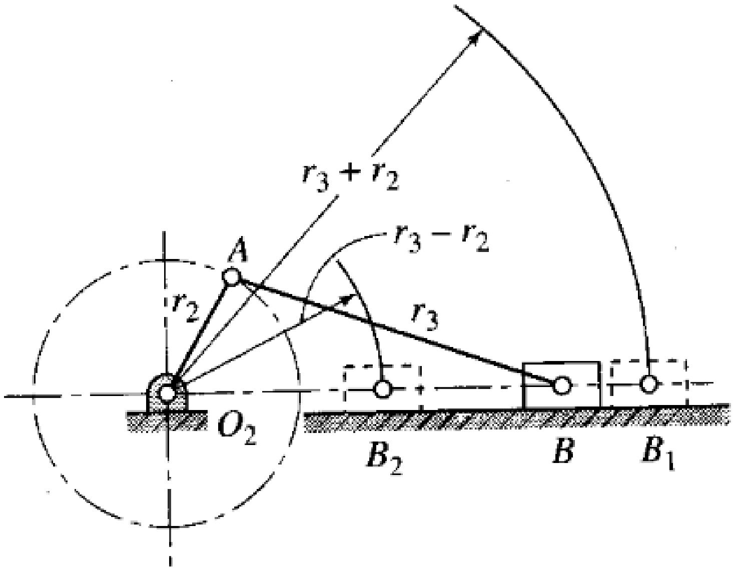

Fig.2.2 – Centered slider-crank mechanism

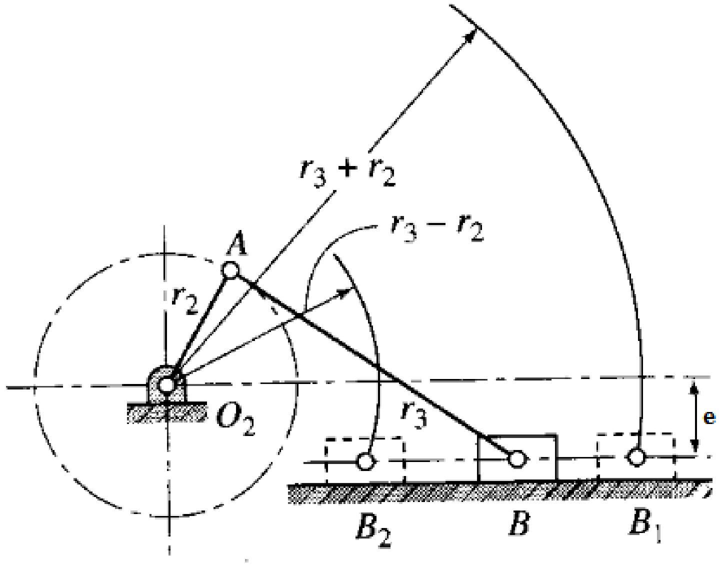

Fig.2.3 – General or offset slider crank mechanism

All points on the coupler of the isosceles slider-crank mechanism generate elliptical paths. The paths generated by the points on the coupler of the slider-crank are not elliptical, but they are always symmetrical about the axis O_2B.

The linkage of general or offset slider-crank mechanism certain special effects can be obtained by changing the offset distance e. Ex. the stroke B_1B_2 is always greater than $2 \times \text{crank radius } r_2$.

This feature can be used to synthesize a quick return mechanism where a slower working stroke is desired. Also, the crank angle required to execute the forward stroke is different from that of the return stroke.

2.4 Two Position Synthesis of Crank and Rocker Mechanism

The limiting positions of the rocker in a crank and rocker mechanism are shown as points B_1 and B_2 (Found same as slider-crank linkage).

In this particular case, the crank executes the angle Ψ while the rocker moves from B_1 to B_2. Note on the return stroke that the rocker swing from B_2 to B_1 through the same angle but the crank moves through the angle (360° – Ψ).

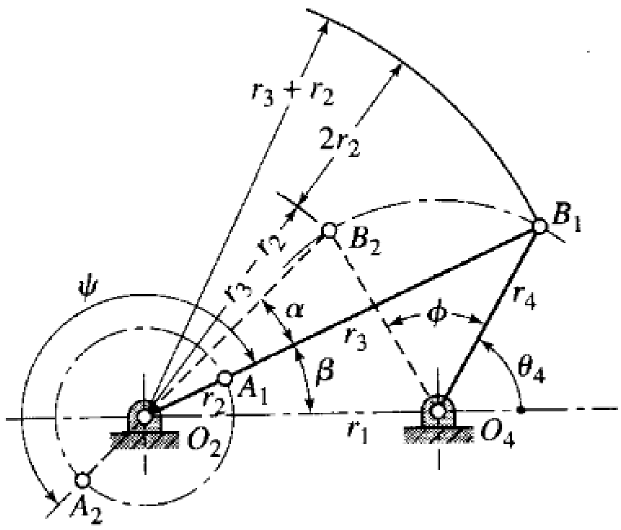

Fig.2.4 - Extreme Position of Crank and Rocker Mechanism

There are many cases in which the crank and rocker mechanism is superior to the cam and follower system. Among the advantages over the cam, the system is smaller forces involved, the elimination of retaining spring, and the closer clearance because of the use of revolute pairs.

Cutting stroke B_2 to B_1 (Ø angle on the rocker) Ψ angle on the crank

Return stroke B_1 to B_2 (Ø angle on the rocker) 360° – Ψ angle on the crank

$$Q = \frac{\psi}{2\pi - \psi} = \frac{180 + \alpha}{180 - \alpha} \qquad \left\{ \begin{array}{c} t_1 = \dfrac{\psi}{\omega} \\ t_2 = \dfrac{360 - \psi}{\omega} = \dfrac{2\pi - \psi}{\omega} \end{array} \right\}$$

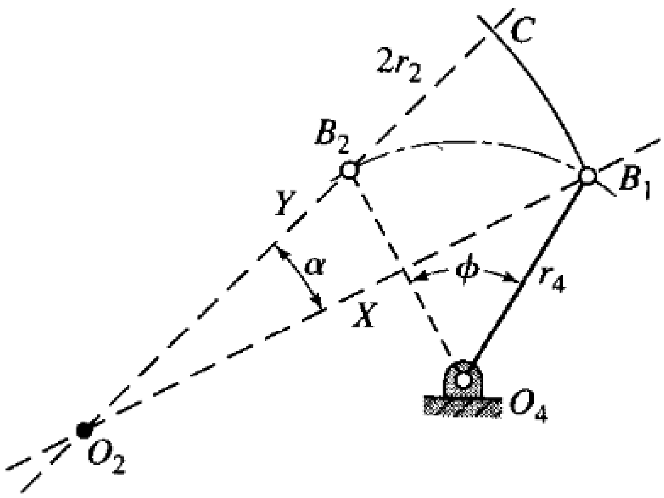

Fig.2.5 - Synthesis of a four-bar linkage to generate rocker angle Ø

To synthesis, a crank and rocker mechanism for a specified value of Ø and α, locate the point O_4 in the figure and choose any desired rocker length r_4, then draw the two positions O_4B_1 and O_4B_2 of link 4 separated by the angle Ø as given.

Through B_1 construct any line *X* Then through B_2 construct the line *Y* at given angle α to line X. The intersection of these two lines defines the location of the crank pivot O_2. Because line X was originally chosen arbitrarily, there is an infinite number of solutions to this problem.

The distance B_2C is $2r_2$ or twice the crank length. So we bisect this distance to find r_2.

2.5 Inversion Method of Synthesis for Four-Bar Mechanism using Three Point

- In the below figure, the motion of input rocker O_2A through the angle ψ_{12} causes the motion of the output rocker O_4B through angle $\emptyset_{12}$.

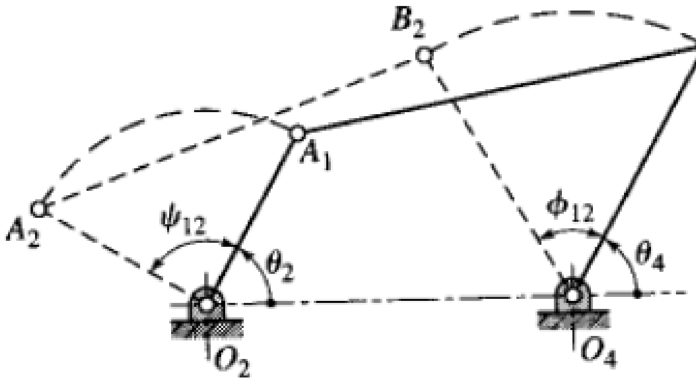

Fig.2.6 - Rotation of input rocker O_2A through the angle ψ_{12} cause rocker O_4B to rock through the angle $\emptyset_{12}$

- To employ inversion as a technique of synthesis, let us hold O_4B stationary and permit the remaining links, including the frame, to occupy the same relative positions.
- The result is called inverting on the output rocker. Note that A_1B_1 is positioned the same in the below figure. Therefore the inversion is made on the O_4B_1 position. Because O_4B_1 is fixed, the frame will have to move in order to get the linkage to the A_2B_2 position. In fact, the frame must move backward through the angle $\emptyset_{12}$. The second position is therefore $O_2'A_2'B_2'O_4$.

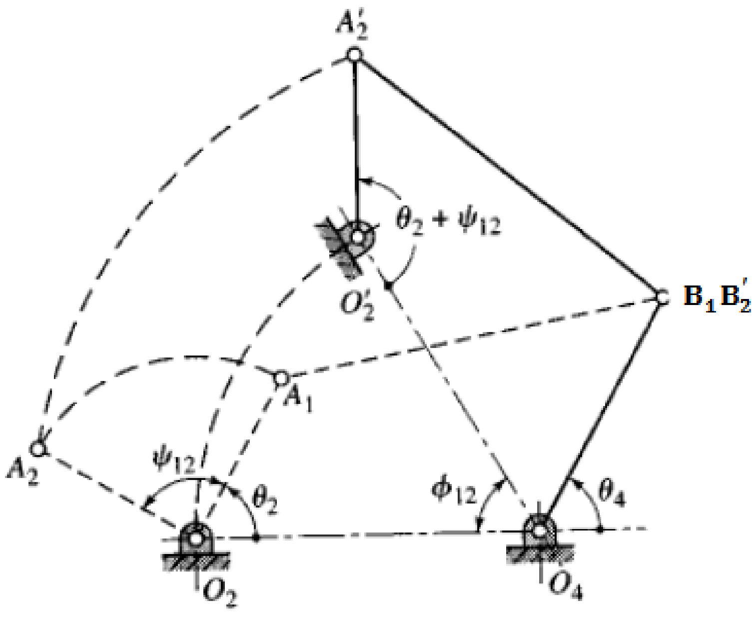

Fig.2.7 - Linkage inverted on the O_4B position

- The below figure illustrates a problem and the synthesized linkage in which it is desired to determine the dimensions of a linkage in which the output lever is to occupy three specified positions corresponding to three given positions of input lever.
- The starting angle of the input lever is θ_2; and ψ_{12}, ψ_{23}, and ψ_{13} are swing angle respectively between the three design positions 1 and 2, 2 and 3, and 1 and 3. Corresponding angles of swing $\emptyset_{12}$, $\emptyset_{23}$ and $\emptyset_{13}$ are desired for the output lever. The length of link 4 and the starting position O_4 are to be determined.

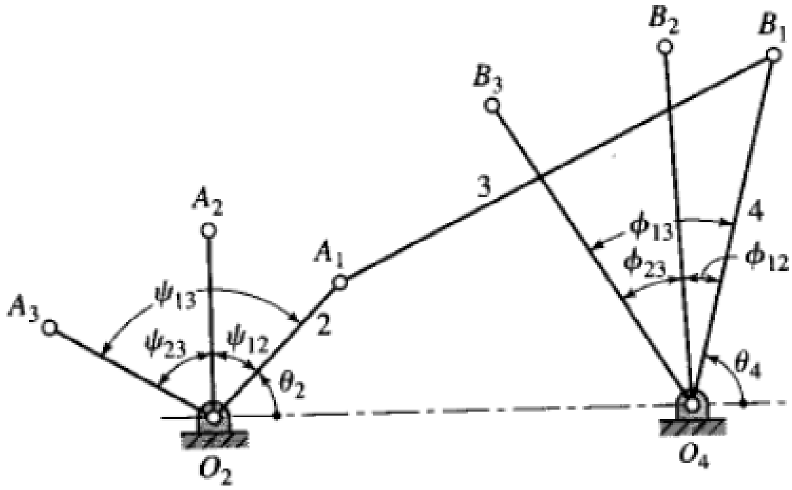

Fig.2.8 – Three positions of input lever and output lever

2.6 Chebychev Spacing for Precision Positions

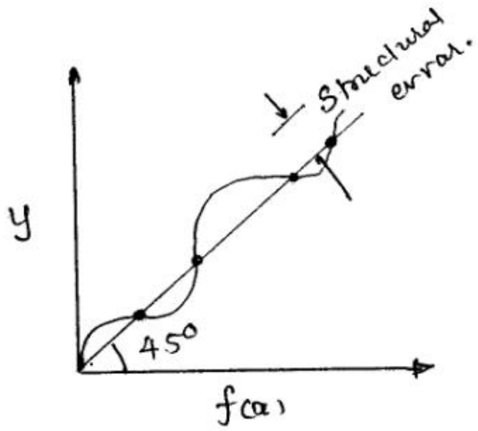

Fig.2.9 - Structural Error

- We need to work with two or three or four positions of the linkage called precision positions and to find a linkage that exactly satisfies the desired function at a few chosen positions.
- Structural error is defined as the theoretical difference between the function produced by the synthesized linkage and the function originally prescribed.
- A very good trial for the spacing of these precision positions is called Chebychev spacing. For n precision position in the range $x_0 \le x \le x_{n+1}$, the ***Chebychev spacing*** according to Freudenstein and Sandor, is

$$x_j = \frac{1}{2}(x_{n+1} + x_0) - \frac{1}{2}(x_{n+1} - x_0)cos\frac{(2j-1)\pi}{2n}$$

Where $j = 1,2, \dots n$ And n = No. of precision positions

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2.7 Problems

Ex. 2.1 **[GTU; June-2016; 7 Marks] [GTU; Jan.-2016; 7 Marks]**

A four-bar mechanism is to be designed, by using three precision points, to generate the function

$$y = x^{1.5}, \text{ for the range } 1 \le x \le 4 .$$

Assuming 30° starting position and 120° finishing position for the input link and 90° starting position and 180° finishing position for the output link, find the values of x, y, θ and φ corresponding to the three precision points.

Solution: Given Data:

$x_S = 1;\ x_F = 4;\ \theta_S = 30°;\ \theta_F = 120°;\ \varphi_S = 90°$ and $\varphi_F = 180°$

The three **values of *x*** corresponding to three precision points (i.e. **for *n* = 3**) according to Chebychev's spacing are given by,

$$\boldsymbol{x_j = \frac{1}{2}(x_F + x_S) - \frac{1}{2}(x_F - x_S)\left(\cos\left[\frac{(2j-1)\pi}{2n}\right]\right)}$$

$$\therefore x_j = \frac{1}{2}(4+1) - \frac{1}{2}(4-1)\left(\cos\left[\frac{(2j-1)\pi}{2(3)}\right]\right)$$

$$\therefore \boldsymbol{x_j = 2.5 - 1.5\left(\cos\left[\frac{(2j-1)\pi}{6}\right]\right)}$$

For *j* = 1,

$$x_1 = 2.5 - 1.5\left(\cos\left[\frac{(2(1)-1)\pi}{6}\right]\right)$$

$$\therefore x_1 = 2.5 - 1.5\left(\cos\left[\frac{\pi}{6}\right]\right)$$

$$\therefore x_1 = 2.5 - 1.5\,(\cos 30°)$$

$$\therefore \boldsymbol{x_1 = 1.2}$$

For *j* = 2,

$$x_2 = 2.5 - 1.5\left(\cos\left[\frac{(2(2)-1)\pi}{6}\right]\right)$$

$$\therefore x_2 = 2.5 - 1.5\left(\cos\left[\frac{3\pi}{6}\right]\right)$$

$$\therefore x_2 = 2.5 - 1.5\,(\cos 90°)$$

$$\therefore \boldsymbol{x_2 = 2.5}$$

For *j* = 3,

$$x_3 = 2.5 - 1.5\left(\cos\left[\frac{(2(3)-1)\pi}{6}\right]\right)$$

$$\therefore x_3 = 2.5 - 1.5\left(\cos\left[\frac{5\pi}{6}\right]\right)$$

$$\therefore x_3 = 2.5 - 1.5\,(\cos 150°)$$

$$\therefore \boldsymbol{x_3 = 3.8}$$

Since y = $x^{1.5}$, therefore the corresponding **values of y** are

$$\boldsymbol{y_S} = (x_S)^{1.5} = (1)^{1.5} = \mathbf{1}$$

$$\boldsymbol{y_1} = (x_1)^{1.5} = (1.2)^{1.5} = \mathbf{1.316}$$

$$\boldsymbol{y_2} = (x_2)^{1.5} = (2.5)^{1.5} = \mathbf{3.952}$$

$$\boldsymbol{y_3} = (x_3)^{1.5} = (3.8)^{1.5} = \mathbf{7.41}$$

$$\boldsymbol{y_F} = (x_F)^{1.5} = (4)^{1.5} = \mathbf{8}$$

The three **values of θ** corresponding to three precision points are given by

$$\theta_j = \theta_S + \frac{\theta_F - \theta_S}{x_F - x_S}(x_j - x_S)$$

$$\therefore \theta_j = 30 + \frac{120 - 30}{4 - 1}(x_j - 1) = 30 + \frac{90}{3}(x_j - 1) = 30 + 30(x_j - 1)$$

For *j* = 1,

$$\therefore \boldsymbol{\theta_1} = 30 + 30(1.2 - 1) = \mathbf{36°}$$

For *j* = 2,

$$\therefore \boldsymbol{\theta_2} = 30 + 30(2.5 - 1) = \mathbf{75°}$$

For *j* = 3,

$$\therefore \boldsymbol{\theta_3} = 30 + 30(1.2 - 1) = \mathbf{114°}$$

The three **values of φ** corresponding to three precision points are given by

$$\varphi_j = \varphi_S + \frac{\varphi - \varphi_S}{y_F - y_S}(y_j - y_S)$$

$$\therefore \varphi_j = 90 + \frac{180 - 90}{8 - 1}(y_j - 1) = 90 + \frac{90}{7}(y_j - 1)$$

For *j* = 1,

$$\therefore \boldsymbol{\varphi_1} = 90 + \frac{90}{7}(1.316 - 1) = \mathbf{94.06°}$$

For *j* = 2,

$$\therefore \boldsymbol{\varphi_2} = 90 + \frac{90}{7}(3.952 - 1) = \mathbf{127.95°}$$

For *j* = 3,

$$\therefore \boldsymbol{\varphi_3} = 90 + \frac{90}{7}(7.41 - 1) = \mathbf{172.41°}$$

Ex. 2.2 [GTU; January-2017; 7 Marks] [GTU; December-2014; 7 Marks]

Design a four-bar mechanism to co-ordinate the input and output angles as follows:
Input angles = 15°, 30°, and 45°;
Output angles = 30°, 40°, and 55°.

Solution: Given Data:

$\theta_1 = 15°$; $\theta_2 = 30°$; $\theta_3 = 45°$; $\varphi_1 = 30°$; $\varphi_2 = 40°$ and $\varphi_3 = 55°$

The **Freudenstein's equation** is given by

$$K_1 \cos \varphi + K_2 \cos \theta + K_3 = \cos(\theta - \varphi)$$

For $\theta_1 = 15°$ and $\varphi_1 = 30°$;

$$K_1 \cos 30 + K_2 \cos 15 + K_3 = \cos(15 - 30)$$

$$\therefore K_1(0.866) + K_2(0.966) + K_3 = 0.966 \cdots\cdots\cdots (i)$$

For $\theta_2 = 30°$ and $\varphi_2 = 40°$;

$$K_1 \cos 40 + K_2 \cos 30 + K_3 = \cos(30 - 40)$$

$$\therefore K_1(0.766) + K_2(0.866) + K_3 = 0.985 \cdots\cdots\cdots (ii)$$

For $\theta_3 = 45°$ and $\varphi_3 = 55°$;

$$K_1 \cos 55 + K_2 \cos 45 + K_3 = \cos(45 - 55)$$

$$\therefore K_1(0.574) + K_2(0.707) + K_3 = 0.985 \cdots\cdots\cdots (iii)$$

Solving the three simultaneous equations (i), (ii) and (iii), we get

$k_1 = 0.905$; $k_2 = 1.01$ and $k_3 = 1.158$

Assuming the length of one of the links, say "*a*" as one unit, we get the length of the other links.

Let us assume, ***a* = 1 unit**,

$$K_1 = \frac{d}{a}$$

$$\therefore d = a\,(K_1) = 1\,(0.905) = 0.905\ units$$

$$K_2 = \frac{d}{c}$$

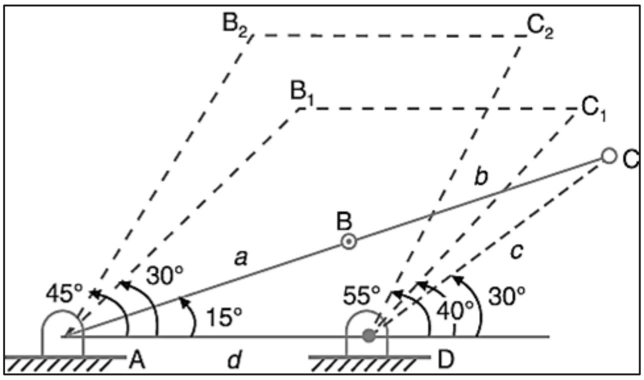
$$\therefore c = \frac{d}{K_2} = \frac{0.905}{1.01} = 0.896\ units$$

$$K_3 = \frac{a^2 - b^2 + c^2 + d^2}{2ac}$$

$$\therefore K_3(2ac) = a^2 - b^2 + c^2 + d^2$$

$$\therefore b^2 = (a^2 + c^2 + d^2) - K_3(2ac)$$

$$\therefore b = 0.74\ units$$

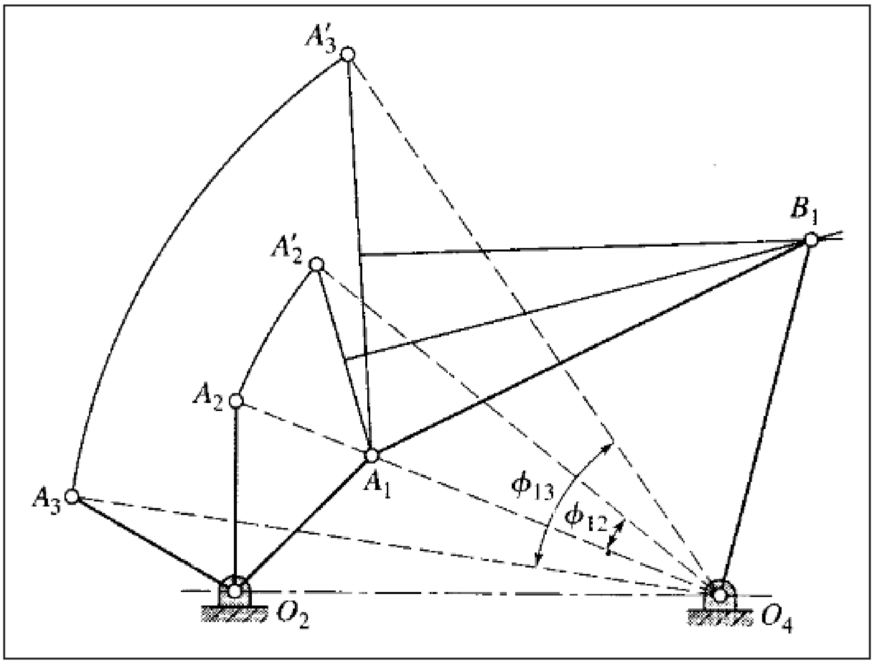


Ex. 2.3 **Synthesize a 4 bar mechanism by the method of inversion for the following specifications.**

$R_{AO_2} = 20 \text{ mm}$ $\quad \psi_{12} = 40°$ $\quad \phi_{12} = 30°$ $\quad \theta_2 = 45°$

$R_{O_4O_2} = 60 \text{ mm}$ $\quad \psi_{23} = 35°$ $\quad \phi_{23} = 25°$

Solution:



- The solution to the problem is given in the figure and is based on inverting the linkage on link 4.
- First, we draw the input rocker O_2A in the three specified positions and locate the desired position for O_4.
- Because we will invert on link 4 in the first design position we draw a ray from O_4 to A_2 and rotate it backward through the angle ϕ_{12} to locate A'_2.
- Similarly, we draw another ray O_4A_3 and rotate it backward through the angle ϕ_{13} to locate A'_3.
- Because we are inverting on the first design position, A_1 and A'_1 are coincident.
- Now we draw mid normals to the line $A_1A'_2$ and $A_1A'_3$. These intersect at B_1 and define the length of coupler link 3 and the length of starting position of link 4.

Ex. 2.4 **Four bar Crank-Rocker quick return linkage for specified time ratio. Time ratio = 1:1.25 with 45° output rocker motion. Design the synthesis.**

Solution:

$$T_R = \frac{\alpha}{\beta} \qquad \alpha + \beta = 360°$$

$$\text{Construction angle } \delta = |180 - \alpha|$$

$$= |180 - \beta|$$

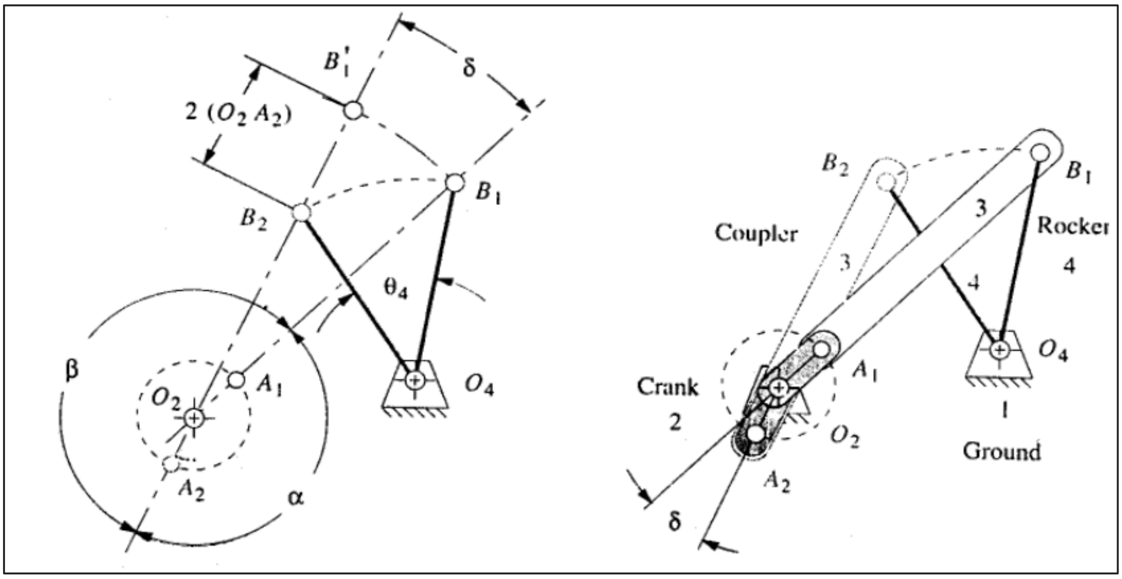
1. Draw the output link O_4B in both extreme positions, in any convenient location, such that the desired angle of motion θ_4, is subtended.
2. Calculate α, β, and δ using equations. In this example, $\alpha = 160°$, $\beta = 200°, \delta = 20°$.
3. Draw a construction line through point B_1 at any convenient angle.
4. Draw a construction line through point B_2 at angle δ from the first line.
5. Label the intersection of the two construction lines O_2.

6. The line O_2O_4 now defines the ground link.
7. Calculate the lengths of crank and coupler by measuring O_2B_1 and O_2B_2 and solve simultaneously.

$$\text{Coupler} + \text{crank} = O_2B_1$$

$$\text{Coupler} - \text{crank} = O_2B_2$$

Or we can construct the crank length by swinging an arc centered at O_2 from B_1 to cut line O_2B_2 extended. Label that intersection B_1'. The line B_2B_1' is twice the crank length. Bisect this line segment to measure crank length O_2A_1.



(a) (b)

(a) Construction of a quick return Grashof crank rocker

(b) The finished linkage in its two toggle positions

References:

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