2

Static Forces on Surface and Buoyancy

Contents

2.1 Fluid Statics

The branch of fluid mechanics associated with the behavior of fluid at rest. Fluid Statics deals with problems associated with fluids at rest. In fluid statics, there is no relative motion between adjacent fluid layers. Therefore, there is no shear stress in the fluid trying to deform it. The only stress in fluid statics is normal stress.

1) Normal stress is due to pressure
2) Variation of pressure is due only to the weight of the fluid → fluid statics is only relevant in presence of gravity fields.

Fluid statics is used to determine the forces acting on floating or submerged bodies and the forces developed by devices like hydraulic presses and car jacks.

The design of many engineering systems such as water dams and liquid storage tanks requires the determination of the forces acting on their surfaces using fluid statics. The complete description of the resultant hydrostatic force acting on a submerged surface requires the determination of the magnitude, the direction, and the line of action of the force.

Applications: Floating or submerged bodies, water dams and gates, liquid storage tanks, etc.

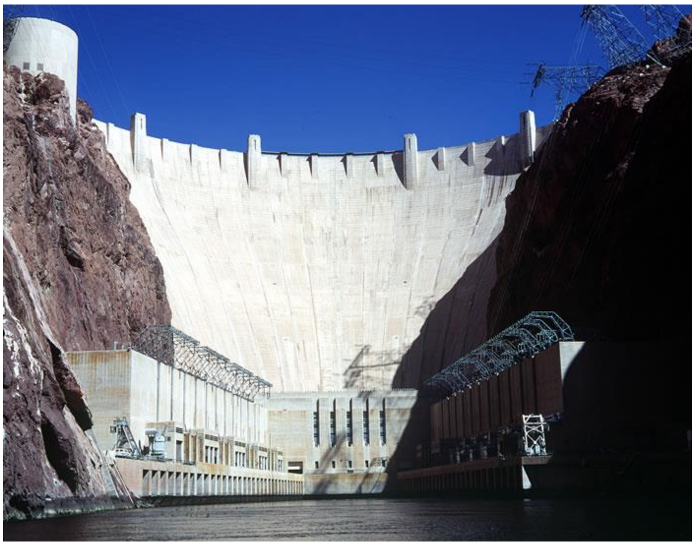

Fig.2.1 Hover Dam

2.2 Action of Fluid Pressure on a Surface

- Since pressure is defined as force per unit area, when fluid pressure p acts on a solid boundary or across any plane in the fluid the force exerted on each small elemen of area δA will be pδA, and, since the fluid is at rest, this force will act at right angles to the boundary or plane at the point under consideration.

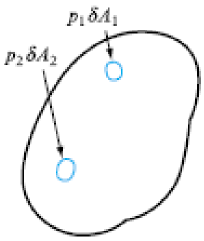

Fig.2.2 Forces on a plane surface

- In a body of fluid, the pressure p may vary from point to point and the forces on each element of area will also vary. If the fluid pressure acts on or across a plane surface, all the forces on the small elements will be parallel shown in *Fig.2.2* and can be represented by a single force, called the resultant force, acting at right angles to the plane through a point called the centre of pressure.

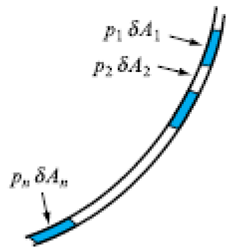

Fig.2.3 Forces on a Curve Surface

$$R = p_1\delta A_1 + p_2\delta A_2 \dots\dots\dots = \sum p\delta A \qquad \text{Eq. (2.1)}$$

- If the boundary is a curved surface, the elementary forces will act perpendicular to the surface at each point and will, therefore, not be parallel as shown in *Fig.2.3*. The resultant force can be found by resolution or by a polygon of forces, but will be less than $\Sigma p\delta A$.
- For example, in the extreme case of the curved surface of a bucket filled with water as shown in *Fig.2.4*. the elementary forces acting radially on the vertical wall will balance and the resultant force will be zero. If this were not so, there would be an unbalanced horizontal force in some direction and the bucket would move of its own accord.

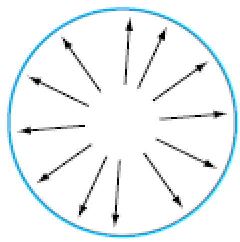

Fig.2.4 Forces on a Cylindrical Surface

2.3 Total Pressure and Centre of Pressure

- Total pressure is defined as the force exerted by a static fluid on a surface either plane or curved when the fluid comes in contact with the surface and this force always normal to the surface.
- Centre of pressure is defined as the point of application of the total pressure on the surface.

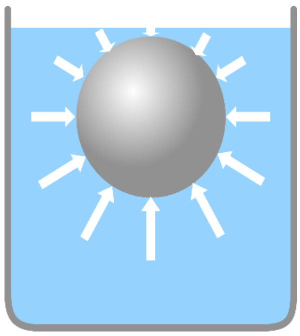

Fig.2.5 Object immersed in fluid

- There are four cases of submerged surfaces on which the total pressure force and centre of pressure is to be determined.
- There are four cases of submerged surfaces on which the total pressure force and centre of pressure is to be determined.
- The submerged surfaces may be:
  1. Vertical plane surface
  2. Horizontal plane surface
  3. Inclined plane surface

4. Curved surface

2.4 Resultant force and centre of pressure on a plane surface immersed in a liquid

2.4.1 Vertical Plane Surface Submerged in Liquid

- Consider a plane surface of arbitrary shape immerged in a liquid as shown in *Fig.2.6*.

– Let, $A =$ Total area of the surface

$\bar{h} =$ Distance of C.G of the area from free surface of liquid

$G =$ Centre of gravity of plane surface

$P =$ Centre of pressure

$h^* =$ Distance of centre of pressure from free surface of liquid

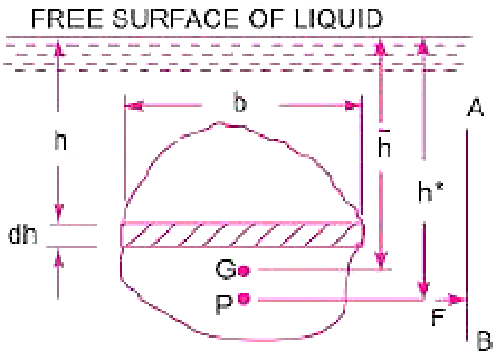

Fig.2.6 Vertical plane surface submerged in liquid

1) Total pressure force (F)

- The total pressure on the surface may be determined by dividing the entire surface into a number of small parallel strips. The force on small srtip is then calculated and the total pressure force on the whole area is calculated by integrating the force on small strip.
- Consider a strip of thickness dh and width b at depth of h from free surface of liquid as in *Fig.2.6*.
- Pressure intensity on the strip, $p = \rho g h$ *Eq. (2.2)*
- Area of strip, $dA = b \times dh$ *Eq. (2.3)*
- Total pressure force on strip, $dF = P \times dA = \rho g h \times b \times dh$ *Eq. (2.4)*
- Total pressure force on whole surface,

$$F = \int dF = \int \rho g h \times b \times dh = \rho g \int b \times \mathrm{h} \times dh \quad \textit{Eq. (2.5)}$$

But, $\int b \times \mathrm{h} \times dh = \int h \times dA =$ Moment of surface area about the free surface of liquid

$=$ Area of surface $\times$ Distance of C.G from free surface $= A \times \bar{h}$

$$F = \rho g A \bar{h} \quad \textit{Eq. (2.6)}$$

2) Position of Centre of Pressure

- Centre of pressure is calculated by using the principle of moments which states that the moment of the resultant force about an axis is equal to the sum of moments of the components about the same axis.
- The resultant force F is acting at P, at distance h^* from free surface of the liquid as shown in *Fig.2.6*.
- Hence moment of the force F about free surface of liquid,

$$= F \times h^* \quad \textit{Eq. (2.7)}$$

- Hence moment of the force dF about free surface of the liquid,

$$= dF \times h$$

$$= \rho g h \times b \times dh \times h \quad (\because dF = P \times dA = \rho g h \times b \times dh)$$

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- Sum of moment of all such forces about free surface of liquid is,

$$= \int \rho g h \times b \times dh \times h$$

$$= \rho g \int b h^2 dh$$

$$= \rho g \int h^2\, dA$$

- But $\int h^2\, dA$ = Second moment of inertia of the surface about free surface of liquid = I_0
- Sum of moment of all such forces about free surface of liquid is,

$$= \rho g \int h^2\, dA = \rho g I_0 \qquad \textit{Eq. (2.8)}$$

- Comparing equations 2.7 & 2.8 We get,

$$F \times h^* = \rho g I_0$$

$$\rho g A\bar{h} \times h^* = \rho g I_0 \qquad (\because F = \rho g A\bar{h})$$

$$h^* = \frac{I_0}{A\bar{h}} \qquad \textit{Eq. (2.9)}$$

- But from theorem of parallel axis,

$$I_0 = I_G + A\overline{h^2}$$

- Substituting I_0 in equation (2.9), we get

$$h^* = \frac{I_G + A\overline{h}^2}{A\bar{h}} = \frac{I_G}{A\bar{h}} + \bar{h} \qquad \textit{Eq. (2.10)}$$

- In equation (2.10) $\bar{h}$ is the distance of C.G. of the area of the vertical surface from free surface of liquid. Hence from above equation it is clear that:
  - Centre of pressure (h^*) lies below the centre of gravity of the vertical surface.
  - The distance of centre of pressure from free surface of liquid is independent of the density of the liquid.

2.4.2 Inclined Plane Surface Submerged in Liquid

- Consider an inclined plane surface of arbitrary shape immerged in a liquid such a way that the surface makes an angle θ with the free surface as shown in *Fig.2.7*.

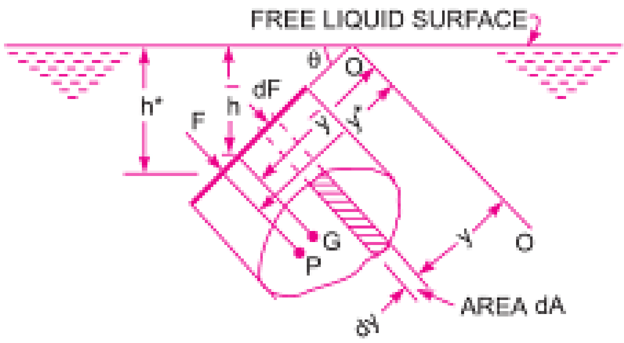


Fig.2.7 Inclined plane surface submerged in liquid

- Let, A = Total area of an inclined surface
  θ = An angle at which the immersed surface is inclined with the liquid surface
  $O - O$ = An axis perpendicular to the plane of surface and plane of surface intersects free liquid surface at O.

G = Centre of gravity of plane surface
P = Centre of pressure
$\bar{y}$ = Distance of C.G. of the surface from O-O
y^* = Distance of Centre of pressure from O-O
h = Depth of small strip from free surface of liquid
$\bar{h}$ = Depth of C.G. of the area from free surface of liquid
h^* = Depth of Centre of pressure from free surface of liquid

1) Total pressure force (F)

- Consider the small strip of area dA at a depth h from the free surface at a distance y from the axis O-O.
- The pressure acting on strip, $p = \rho g h$
- The pressure force acting on strip is, $dF = p \times dA = \rho g h \times dA$
- Total pressure force on whole area, $F = \int dF = \int \rho g h \times dA$ Eq. (2.11)
- From figure, $\frac{h}{y} = \frac{\bar{h}}{\bar{y}} = \frac{h^*}{y^*} = \sin\theta$ Eq. (2.12)
- From equation (2.12), $h = y\sin\theta$

$$F = \int dF = \int \rho g \times y\sin\theta \times dA \quad \text{Eq. (2.13)}$$

$$F = \rho g \sin\theta \int ydA \quad \text{Eq. (2.14)}$$

But, $\int ydA = A\bar{y}$ = First moment of area under surface about O-O axis.

$$F = \rho g \sin\theta \times A\bar{y} \quad \text{Eq. (2.15)}$$

$$F = \rho g A\bar{h} \qquad (\because\ \bar{y}\sin\theta = \bar{h}) \quad \text{Eq. (2.16)}$$

- From equations (2.6) & (2.16) , one can conclude that the total pressure force exerted by a static liquid on an inclined plane submerged surface is same as the force exerted on a vertical plane surface as long as the depth of centre of gravity of the surface is unaltered.

2) Position of Centre of Pressure (h*)

- The pressure force acting on strip, $dF = p \times dA = \rho g h \times dA$

$$dF = \rho g\, y\sin\theta \times dA \qquad (\because\ y\sin\theta = \mathrm{h})$$

- The moment of pressure force dF about axis O-O,

$$= dF \times y$$

$$= \rho g\, y\sin\theta \times dA \times y$$

$$= \rho g\ \sin\theta \times y^2 \times dA$$

- Sum of moment of total pressure force about axis O-O,

$$= \int \rho g\ \sin\theta \times y^2 \times dA \quad \textit{Eq. (2.17)}$$

$$= \rho g\ \sin\theta \int y^2 \times dA \quad \textit{Eq. (2.18)}$$

But, $\int y^2 dA$ = Moment of Inertia of surface about O-O $= I_0$

- From equation (2.18),

$$= \rho g\ \sin\theta \times I_0 \quad \textit{Eq. (2.19)}$$

- Moment of total pressure force F about axis O-O is also given by,

$$= F \times y^* \qquad \text{Eq. (2.20)}$$

- Comparing equations (2.19) and (2.20),

$$F \times y^* = \rho g \sin\theta \times I_0 \qquad \text{Eq. (2.21)}$$

$$y^* = \frac{\rho g \sin\theta \times I_0}{F} \qquad \text{Eq. (2.22)}$$

- From equation (2.12) & (2.16),

$$y^* = \frac{h^*}{\sin\theta} \text{ and } F = \rho g A\bar{h} \qquad \text{Eq. (2.23)}$$

- According to parallel axis theorem,

$$I_0 = I_G + A\bar{y}^2 \qquad \text{Eq. (2.24)}$$

- Substituting value of equations (2.23) & (2.24) in equation (2.22),

$$\frac{h^*}{\sin\theta} = \frac{\rho g \sin\theta \times [I_G + A\bar{y}^2]}{\rho g A\bar{h}} \qquad \text{Eq. (2.25)}$$

$$h^* = \frac{\sin^2\theta}{A\bar{h}} \times [I_G + A\bar{y}^2] \qquad \text{Eq. (2.26)}$$

- From equation (2.12),

$$\bar{y} = \frac{\bar{h}}{\sin\theta} \qquad \text{Eq. (2.27)}$$

$$h^* = \frac{\sin^2\theta}{A\bar{h}} \times [I_G + A\frac{\bar{h}^2}{\sin^2\theta}] \qquad \text{Eq. (2.28)}$$

$$h^* = \frac{I_G \sin^2\theta}{A\bar{h}} + \bar{h} \qquad \text{Eq. (2.29)}$$

- Equation (2.29) shows that the location of the centre of pressure for an inclined plane submerged surface depends on the inclination angle.

2.4.3 Horizontal Plane Surface Submerged in Liquid

- Consider a plane horizontal surface immersed in a static fluid is shown in *Fig.2.8*. As every point of the surface at the same depth from the free surface of the liquid, the pressure intensity will be equal on the entire surface is given by $p = \rho g h$. Where h is depth of surface.

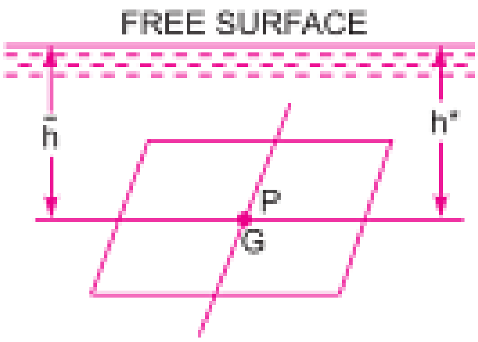


Fig.2.8 Horizontal plane surface submerged in liquid

Let, A = Total area of the surface

G = Centre of gravity of plane surface

P = Centre of pressure

$\bar{h}$ = Depth of C.G. from free surface of liquid

$h^* =$ Depth of Centre of pressure from free surface of liquid

- Total force F on surface,

$$F = \rho g A\bar{h} \quad \text{Eq. (2.30)}$$

2.4.4 Curved Surface Submerged in Liquid

- Consider a curved surface AB submerged in static fluid as shown in *Fig.2.9*. Let dA is the area of small strip at a depth of h form water surface.
- Then Pressure intensity on the area dA, $p = \rho gh$
- The pressure force which is acting normal to the curved surface,

$$dF = p \times dA = \rho gh \times dA \quad \text{Eq. (2.31)}$$

- Total pressure force on the curved surface,

$$F = \int dF = \rho gh \times dA \quad \text{Eq. (2.32)}$$

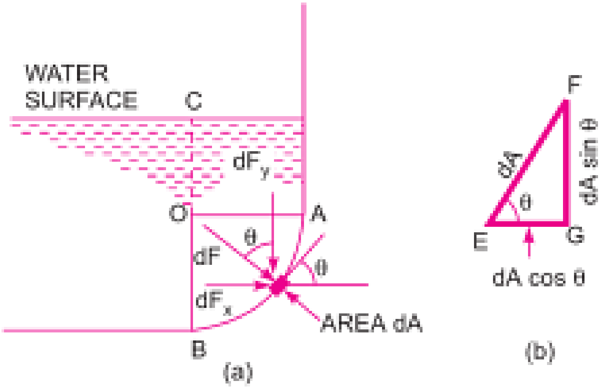


Fig.2.9 Curved surface submerged in liquid

- But here as the direction of the forces on the small area are not in the same direction but varies from point to point. Thus integration of equation (2.31) for curved surface is impossible. Thus resolving the force dF in to two components dF_x and dF_y in the x and y direction respectively. The total force in x and y direction, F_x and F_x are obtaining by integrating dF_x and dF_y.
- Then total pressure forces on the curved surface is,

$$F = \sqrt{{F_x}^2 + {F_y}^2} \quad \text{Eq. (2.33)}$$

- Inclination of resultant with horizontal,

$$\emptyset = \tan^{-1}\left(\frac{F_y}{F_x}\right) \quad \text{Eq. (2.34)}$$

- Now resolving the force dF given by equation (2.31) in x and y direction,

$$dF_x = dF \sin\theta = \rho ghdA \sin\theta \quad \text{Eq. (2.35)}$$

$$dF_y = dF \cos\theta = \rho ghdA \cos\theta \quad \text{Eq. (2.36)}$$

- Total pressure force acting in x and y direction are

$$F_x = \int dF_x = \int dF \sin\theta = \int \rho ghdA \sin\theta = \rho g \int hdA \sin\theta \quad \text{Eq. (2.37)}$$

$$F_y = \int dF_y = \int dF \cos\theta = \int \rho ghdA \cos\theta = \rho g \int hdA \cos\theta \quad \text{Eq. (2.38)}$$

- Fig. 2.9 (b) shows the enlarge area dA. From ΔEFG,

$$EF = dA \qquad Eq.\ (2.39)$$

$$FG = dA \sin\theta \qquad Eq.\ (2.40)$$

$$EG = dA \cos\theta \qquad Eq.\ (2.41)$$

- Thus in equation (2.37), $dA \sin\theta = FG$ = Vertical projection of the area dA and hence the expression $F_x = \rho g \int hdA \sin\theta$ represents the total pressure force on the projected area of curved surface on the vertical plane.
- Also $dA \cos\theta = EG$ = horizontal projection of the area dA and hence the expression $hdA \cos\theta$ is the volume of the liquid contained in the elementary area dA upto free surface of the liquid. Thus $\int hdA \cos\theta$ is the total volume contained between the curved surface extended upto free surface.
- Hence $\rho g \int hdA \cos\theta$ is the total weight of liquid supported by curved surface. Thus,

$$F_y = \rho g \int hdA \cos\theta \qquad Eq.\ (2.42)$$

- F_y =Weight of liquid suppoted by the curved surface upto free surface of liquid

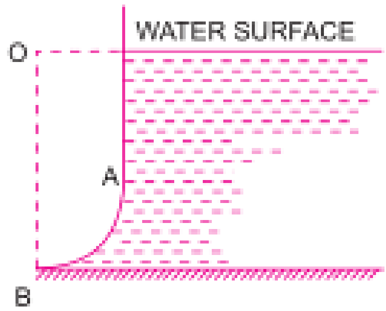
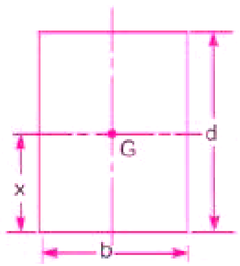
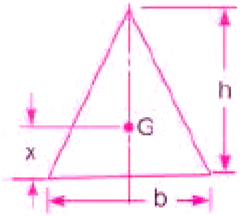
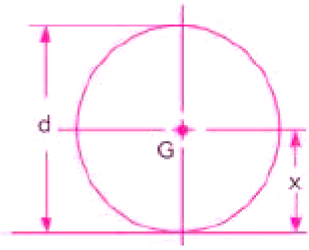
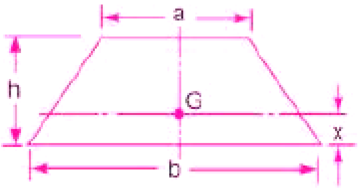


Fig.2.10 Curved Surface not supporting any fluid

- In Fig. 2.10 curved surface not supporting any fluid. In such cases, F_y is equal to the weight of the imaginary liquid supported by AB upto free surface of the liquid. The direction of F_y will be taken in upward direction.

Table 2.1 - The moment of inertia and other geometric properties of some plane surfaces

Plane Surface	**C.G. from the base**	**Area**	**Moment of inertia about an axis passing through C.G.**
	$X = \frac{d}{2}$	$A = b \times d$	$I_G = \frac{bd^3}{12}$

	$X = \frac{h}{3}$	$A = \frac{1}{2} b \times h$	$I_G = \frac{bh^3}{36}$
	$X = \frac{d}{2}$	$A = \frac{\pi}{4} \times d^2$	$I_G = \frac{\pi d^4}{64}$
	$X = \left(\frac{2a+b}{a+b}\right) \times \frac{h}{3}$	$A = \frac{h}{4} \times (a+b)$	$I_G = \left[\frac{a^2+4ab+b^2}{36(a+b)}\right] \times h^3$

2.5 Introduction of Buoyancy

- Objects feels lighter and weighs less in a liquid that it does in air. This observation suggest that a fluid exerts an upward force on a body immerged in it. This force that tends to lift the body is called buoyancy force.

2.5.1 Buoyancy and Centre of Buoyancy

- When a body is immersed in a fluid an upward force exerted by the fluid on the body. This upward force is equal to the weight of the fluid displaced by the body and is called the force of buoyancy.
- It is defined as the point through which the force of buoyancy is supposed to be act. As the force of buoyancy is a vertical force is equal to the weight of the fluid displaced by the body, the centre of buoyancy will be the centre of gravity of the fluid displaced.

2.6 Meta-Centre and Meta Centric Height

2.6.1 Meta-Centre

- It is defined as the point about which a body starts oscillating when the body is tilted by a small angle. The Meta centre may also be defined as the point at which the line of action of the force of buoyancy will meet the normal axis of the body when the body is given a small angular displacement.

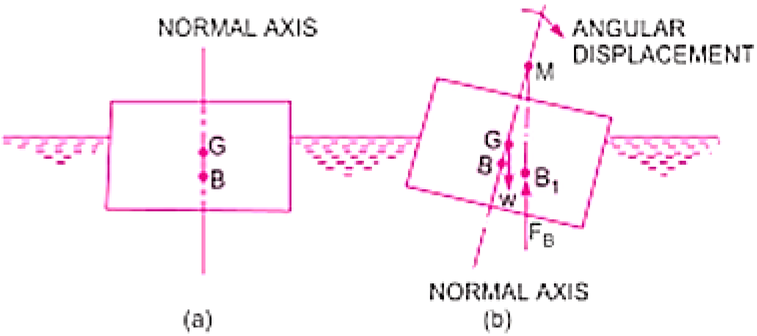


Fig.2.11 Meta Centre

- Consider a body floating in a liquid as shown in *Fig.2.11* (a). Let the body is in equilibrium and G is the centre of gravity and B the centre of buoyancy. For equilibrium, both the points lie on the normal axis, which is vertical.
- Let the body is given a small angular displacement in the clockwise direction as shown in *Fig.2.11*(b). The centre of buoyancy, which is the centre of gravity of the displaced liquid or centre of gravity of the portion of the body submerged in liquid, will now be shifted towards right from the normal axis. Let it is at B_1. The line of action of the force of buoyancy in new position, will intersect the normal axis of the body at some point say M. This point is called meta centre.

2.6.2 Meta Centric Height

- It is the distance MG between the meta centre of the floating body and the centre of gravity of the body.

2.7 Condition for Stability of Submerged Body

- The position of centre of gravity and centre of buoyancy in case of a completely sub-merged body are fixed.

(a) Stable Equilibrium

- Consider a balloon which is completely submerged in air. Let lower portion of the balloon contained heavy material, so that its centre of gravity is lower than centre of buoyancy.
- Let the weight of balloon is W acting through G vertically downwards direction while the buoyant force F_B is acting vertically up through B. For the equilibrium of the balloon $W = F_B$. Now if the balloon is given an angular displacement in the clockwise direction as shown in *Fig.2.12* (a). Then W and F_B constitute a couple acting in the anti-clockwise direction and brings the balloon in the original position. Thus the balloon in this position is in stable equilibrium.
- Thus if $W = F_B$ and point B is above G , the body is said to be in stable equilibrium.

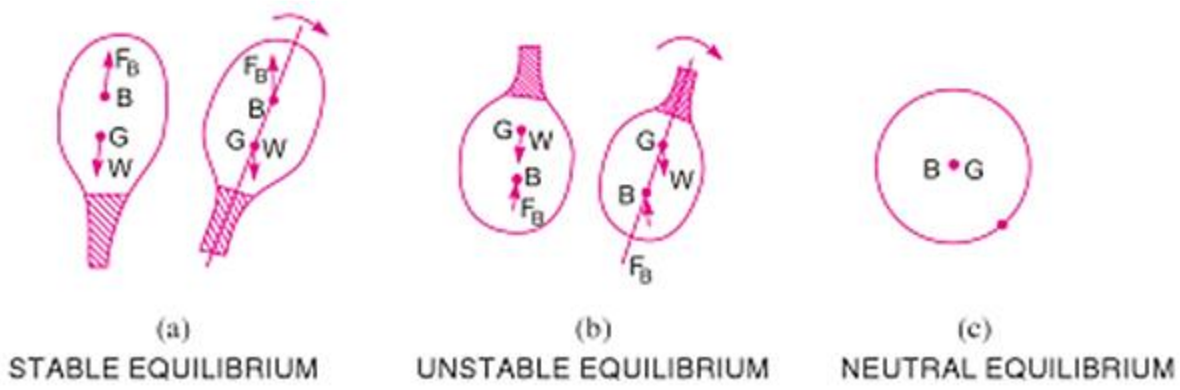


Fig.2.12 Stabilities of Sub-Merged Bodies

(b) Unstable Equilibrium

- If W=F_B and point B is below G, the body is said to be in unstable equilibrium as shown in *Fig.2.12* (b). If the slight angular displacement given to the balloon in the clockwise direction, then W and F_B constitute a couple also acting in the clockwise direction. Thus the balloon does not return to its original position and it is in unstable equilibrium.

(C) Neutral Equilibrium

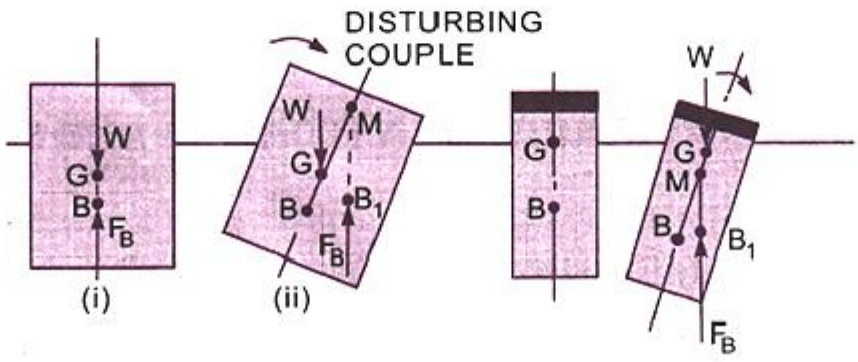
- If W=F_B and point B and G are at the same point as shown in *Fig.2.12* (c), then the body is said to be in neutral equilibrium.

2.8 Condition for Stability of Floating Body

- The stability of a floating body is determined from the position of Meta-centre. In case of floating body, the weight of the body is equal to weight of the liquid displaced.

(a) Stable Equilibrium

- If the point M is above G, the floating body will be in stable equilibrium as shown in *Fig.2.13* (a). If the slight angular displacement given to the balloon in the clockwise direction, the centre of buoyancy sifted from B to B_1 such that the vertical line through BB_1 cuts at M. Then the buoyant force F_B through B_1 and weight W through G constitute a couple acting in the anticlockwise direction. Thus bringing the floating body in original position.



(a) Stable equilibrium M is above G (b) Unstable equilibrium M is below G.

Fig.2.13 Stabilities of Floating Bodies

(b) Unstable Equilibrium

- If the point M is below G, the floating body will be in unstable equilibrium as shown in *Fig.2.13* (b). If the slight angular displacement given to the balloon in the clockwise direction. Then the couple due to buoyant force F_B and weight W constitute a couple also acting in the clockwise direction and thus overturning the floating body.

(c) Neutral Equilibrium

- If the point M is centre of gravity of the body, the floating body will be in neutral equilibrium as shown in *Fig.2.13* (c).

2.9 Experimental Method for Meta Centric Height

- The meta-centric height of the floating vessel can be determined, provided that we know centre of gravity of the floating vessel. Let w_1 is known weight placed over the centre of the vessel as shown in *Fig.2.14* (a).

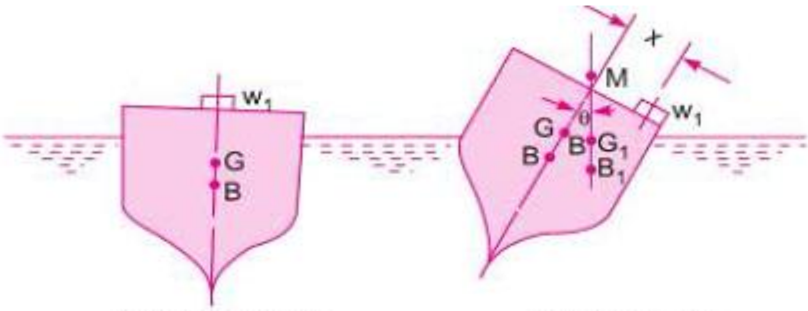


Fig.2.14 Meta Centric Height of The Floating Body

Let W = Weight of the Vessel Including w_1

G = Centre of Gravity of the Vessel

B = Centre of Buoyancy of the Vessel

- The weight w_1 is moved across the vessel towards right through a distance x as shown in *Fig.2.14* (b). The vessel will be tilted. The angle θ is measured by means of a plumbline and a protractor attached on the vessel. The new centre of gravity will shifted to G_1 as the weight has been moved towards the right. Also the centre of buoyancy will change to B_1 as the vessel has tilted. Under equilibrium, the moment cause by movement of the load w_1 through a distance x must be equal to the moment caused by the shift of the centre of gravity from G to G_1.
- The moment due to change of, $G = GG_1 \times W = W \times GM \ \tan\theta$
- The moment due to movement of, $w_1 = w_1 \times x$

$$w_1 x = WGM \tan\theta \qquad \text{Eq. (2.43)}$$

$$GM = \frac{w_1 x}{W \tan\theta} \qquad \text{Eq. (2.44)}$$