3

Motion of Fluid Particles and Streams

Contents

3.1 Introduction to Fluid Kinematics and Fluid Flow

Fluid kinematics deals with describing the motion of fluids without considering the forces and moments that cause the motion.

When the fluid flows over a surface or any boundary, the velocity of the fluid in contact with the boundary must be the same as that of the boundary, and a velocity gradient is created at right angles to the boundary. The shear stresses are developed when the fluid is in motion. The resulting motion is not easily analyzed mathematically, and it is often necessary to enhance theory by experiment.

3.2 Types of Fluid Flow

The fluid flows can be:

1. Steady and Unsteady
2. Uniform and Non-uniform
3. Laminar and Turbulent
4. Compressible and Incompressible
5. Rotational and Irrotational
6. One, Two and Three Dimensional

3.2.1 Steady and Unsteady Flow

Steady Flow	Unsteady Flow
▸ The flow in which fluid characteristics like velocity, acceleration, pressure or density **do not change with time** at any point in the fluid is called steady flow. ▸ Mathematically, $\frac{\partial p}{\partial t} = 0, \quad \frac{\partial v}{\partial t} = 0$ ▸ **Example:** Flow of water with a constant discharge through a pipe.	▸ The flow in which fluid characteristics like velocity, acceleration, pressure or density **change with time** at any point in the fluid is called unsteady flow. ▸ Mathematically, $\frac{\partial p}{\partial t} \neq 0, \quad \frac{\partial v}{\partial t} \neq 0$ ▸ **Example:** Flow of water with a varying discharge through a pipe.

3.2.2 Uniform and Non-uniform Flow

Uniform Flow	Non-uniform Flow
▸ The flow in which velocity of the flow at a given time **does not change with respect to space** is called uniform flow. ▸ Mathematically, $\left(\frac{\partial v}{\partial s}\right)_{t=constant} = 0$ ▸ **Example:** A flow through a constant diameter pipe.	▸ The flow in which velocity of the flow at a given time **changes with respect to space** is called non-uniform flow. ▸ Mathematically, $\left(\frac{\partial v}{\partial s}\right)_{t=constant} \neq 0$ ▸ **Example:** A flow through a pipe having a varying cross-section.

3.2.3 Laminar and Turbulent Flow

Laminar Flow	Turbulent Flow
▶ Laminar flow is one in which the fluid particles moves along well-defined paths or streamlines. ▶ There is no movement of fluid particles from one layer to another. ▶ It is also known as streamline flow or viscous flow where the velocity of flow is very small. ▶ **Examples:** Flow of blood in small veins, Oil flow in bearings, etc.	▶ A turbulent flow is one in which the fluid particles moves in a zig-zag way. ▶ The fluid particles do not move along the well-defined path and cross from one layer to another. ▶ Thus eddies formation takes place which is responsible for high energy loss. ▶ **Examples:** Flow-through river or canals, Smoke from a chimney, etc.

3.2.4 Compressible and Incompressible Flow

Compressible Flow	Incompressible Flow
▶ The flow in which the **density of fluid does not remain constant** during the flow is called compressible flow. ▶ Gases are highly compressible. ▶ **Examples:** Flow of gases through an orifice, nozzle, turbine, compressor, etc.	▶ The flow in which the **density of fluid remains constant** during the flow is called incompressible flow. ▶ Most of the liquids are incompressible. ▶ **Examples:** Flow of liquids through an orifice, nozzle, turbine, pump, etc.

3.2.5 Rotational and Irrotational Flow

Rotational Flow	Irrotational Flow
▶ The rotational flow is a flow in which the fluid particles while flowing along streamlines, also rotates about their own axis. ▶ **Example:** Motion of liquid in a rotating cylinder.	▶ The irrotational flow is a flow in which the fluid particles while flowing along streamlines, do not rotates about their own axis. ▶ **Example:** Flow of water in emptying washbasin.

3.2.6 One, Two and Three Dimensional Flow

1-D Flow	2-D Flow	3-D Flow
▶ One-dimensional flow is a flow in which the velocity of the flow is a function of time and one space coordinate $(x, y\ or\ z)$. ▶ Mathematically, $u = f(x,t)$ $v = 0$ $w = 0$ ▶ **Example:** The flow through a pipe.	▶ Two-dimensional flow is a flow in which the velocity of the flow is a function of time and two space coordinates $(xy, xz\ or\ yz)$. ▶ Mathematically, $u = f_1(x,y,t)$ $v = f_2(x,y,t)$ $w = 0$ ▶ **Example:** The flow between two parallel plates.	▶ Three-dimensional flow is a flow in which the velocity of the flow is a function of time and three space coordinates (x, y, z). ▶ Mathematically, $u = f_1(x,y,z,t)$ $v = f_2(x,y,z,t)$ $w = f_3(x,y,z,t)$ ▶ **Example:** Flow in converging or diverging pipe section.

3.3 Methods of Describing the Fluid Flow

During fluid motion, each fluid particle has its own velocity and acceleration at any point of time. Velocity and acceleration of a particle may change with ***time or position***. Two methods are used in describing the fluid motion (1) Langrangian Method and (2) Eulerian Method.

In the **Langrangian method**, the observer follows a single fluid particle during its motion and will observe the change in its properties like velocity, acceleration, density, etc.

In the **Eulerian method**, the observer concentrates on a fixed point or a region and will observe the change in velocity, acceleration, density, etc. at that point only.

3.3.1 Difference between the Langrangian and the Eulerian Approach

*Table **3.1** - Difference Between Langrangian and Eulerian Approach*

Sr. No.	Langrangian Method	Eulerian Method
1	Observer concentrates on the movement of a single fluid particle	An observer concentrates on the fixed point particles
2	An observer has to move with the fluid particle to observe its movement.	An observer remains stationary and observes changes in the fluid parameters at the fixed point only
3	The path & changes in velocity, acceleration, pressure and density of a single particle are described	The method describes the overall flow characteristics at various points as fluid particles pass
4	Not commonly used	Commonly used

3.4 Discharge and Conservation of Mass

<u>Discharge:</u>

Discharge is defined as a quantity of fluid flowing per second through a section of pipe or a channel.

For liquids, the units of discharge (Q) are m^3/sec or lit/sec. For gases, the units are kgf/sec or N/sec.

Mathematically, Discharge is given by,

$$Q = \frac{Volume}{Time} = \frac{Area \times Length}{Time} = Area \times Velocity = A \times v$$

<u>Conservation of Mass Principle or Continuity Equation:</u>

The principle of conservation of mass states that mass cannot be created nor destroyed.

For a control volume,

$$Rate\ at\ which\ mass\ enters\ the\ C.V. - Rate\ at\ which\ mass\ leaves\ the\ C.V.$$
$$= Rate\ of\ accumulation\ of\ mass\ in\ the\ C.V.$$

$$\therefore \dot{m}_{in} - \dot{m}_{out} = \frac{dm_{CV}}{dt}$$

For steady flow, $\frac{dm_{CV}}{dt} = 0$

$$\therefore \dot{m}_{in} = \dot{m}_{out}$$

For 1-D, steady flow,

$$\dot{m} = \rho Q = \rho A v = Constant$$

$$\therefore \dot{m} = \rho_1 A_1 v_1 = \rho_2 A_2 v_2 = Constant$$

For incompressible flow,

$$\rho = constant$$

$$\therefore A_1 v_1 = A_2 v_2 = Constant$$

3.5 Continuity Equation for 3-D Flow in Cartesian Coordinate System

- Consider a control volume or fluid element of lengths $dx, dy\ and\ dz$ in the $x, y, and\ z$ direction respectively. Let $u, v\ and\ w$ are the inlet velocity components in $x, y, and\ z$ directions respectively.

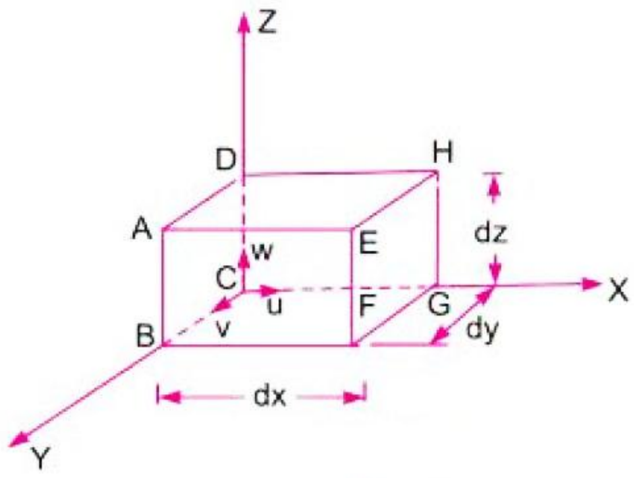

Fig.3.1 – 3-Dimensional Fluid Element

- Mass of fluid entering the face ABCD per second,

$$\dot{m}_{in_x} = Density \times Velocity\ in\ x\ direction \times area\ of\ face\ ABCD$$

$$\therefore \dot{m}_{in_x} = \rho \times u \times (dy \times dz) \qquad \text{Eq. (3.1)}$$

- Mass of fluid leaving the face EFGH per second,

$$\dot{m}_{out_x} = \dot{m}_{in_x} + \frac{\partial(\dot{m}_{in_x})}{\partial x} dx$$

$$\therefore \dot{m}_{out_x} = \rho u dy dz + \frac{\partial(\rho u dy dz)}{\partial x} dx$$

$$\therefore \dot{m}_{out_x} = \rho u dy dz + \frac{\partial(\rho u)}{\partial x} dx dy dz \qquad \text{Eq. (3.2)}$$

- The gain of mass in x – direction,

$$Gain\ of\ mass\ in\ x - direction = \dot{m}_{in_x} - \dot{m}_{out_x}$$

$$\therefore Gain\ of\ mass\ in\ x - direction = \rho u dy dz - \rho u dy dz - \frac{\partial(\rho u)}{\partial x} dx dy dz$$

$$\therefore Gain\ of\ mass\ in\ x - direction = -\frac{\partial(\rho u)}{\partial x} dx dy dz \qquad \text{Eq. (3.3)}$$

Similarly,

- The gain of mass in y – direction,

$$Gain\ of\ mass\ in\ y - direction = \dot{m}_{in_y} - \dot{m}_{out_y}$$

$$\therefore Gain\ of\ mass\ in\ y - direction = \rho v dxdz - \rho v dxdz - \frac{\partial(\rho v)}{\partial y} dxdydz$$

$$\therefore Gain\ of\ mass\ in\ y - direction = -\frac{\partial(\rho v)}{\partial y} dxdydz \qquad \text{Eq. (3.4)}$$

And,

- The gain of mass in z – direction,

$$\therefore Gain\ of\ mass\ in\ z - direction = -\frac{\partial(\rho w)}{\partial z} dxdydz \qquad \text{Eq. (3.5)}$$

Therefore,

- Net gain of masses,

$$\therefore Net\ gain\ of\ masses = -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] dxdydz \qquad \text{Eq. (3.6)}$$

- Since the mass is neither created nor destroyed, the net increase of mass must be equals to the rate of increase of mass of fluid in the element.
- But, the mass of fluid in the element is $\rho dxdydz$, and its rate of increase with time is,

$$= \frac{\partial(\rho dxdydz)}{\partial t}$$

$$= \frac{\partial \rho}{\partial t} dxdydz \qquad \text{Eq. (3.7)}$$

- For a control volume or fluid element,

$$Net\ gain\ of\ masses = Rate\ of\ increase\ of\ mass\ in\ the\ control\ volume$$

$$\therefore -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] dxdydz = \frac{\partial \rho}{\partial t} dxdydz$$

$$\therefore \boldsymbol{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0} \qquad \text{Eq. (3.8)}$$

- *Eq. (3.8)* is the continuity equation in cartesian coordinates in its most general form. This equation applies to:
    1) Steady and Unsteady flow
    2) Uniform and Non-uniform flow
    3) Compressible and Incompressible flow

- **For Steady flow,**

$$\frac{\partial \rho}{\partial t} = 0$$

$$\therefore \boldsymbol{\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0} \qquad \text{Eq. (3.9)}$$

- **If the fluid is incompressible**, then $\rho = constant$

$$\therefore \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \text{Eq. (3.10)}$$

- **For a 2-D flow,** $w = 0$

$$\therefore \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \text{Eq. (3.11)}$$

3.6 Velocity and Acceleration

Let V is the resultant velocity at any point in a fluid flow. Let u, v and w are its components in x, y and z directions. The velocity components are the functions of space coordinates and time.

Mathematically the velocity components are given as,

$$u = f_1(x, y, z, t)$$
$$v = f_2(x, y, z, t)$$
$$w = f_3(x, y, z, t)$$

Resultant velocity is given by,

$$V = ui + vj + wk$$

$$\therefore V = \sqrt{u^2 + v^2 + w^2} \qquad \text{Eq. (3.12)}$$

Let a_x, a_y and a_z are the total acceleration in x, y and $z$directions respectively. Then by the ***chain rule of differentiation***, we have,

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} + \frac{\partial u}{\partial t}$$

But,

$$\frac{dx}{dt} = u, \qquad \frac{dy}{dt} = v, \qquad \frac{dz}{dt} = w$$

$$\therefore a_x = \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \qquad \text{Eq. (3.13)}$$

Similarly,

$$\therefore a_y = \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \qquad \text{Eq. (3.14)}$$

And,

$$\therefore a_z = \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} \qquad \text{Eq. (3.15)}$$

Resultant acceleration is given by,

$$A = a_x i + a_y j + a_z k$$

$$\therefore A = \sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2} \qquad \text{Eq. (3.16)}$$

3.6.1 Local Acceleration

Local acceleration is defined as the rate of increase of velocity with respect to time at a given point in a flow field. In *Eq. (3.13)* to *Eq. (3.15)*, the terms $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$ or $\frac{\partial w}{\partial t}$ is known as local acceleration.

3.6.2 Convective Acceleration

Convective acceleration is defined as the rate of change of velocity due to the change of position of fluid particles in a fluid flow. In *Eq. (3.13)* to *Eq. (3.15)*, the terms other than $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$ or $\frac{\partial w}{\partial t}$ are known as convective acceleration.

- **For Steady flow,**

  For steady flow, $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$ and $\frac{\partial w}{\partial t}$ are zero, therefore local acceleration is zero but convective acceleration is not necessarily zero.

- **For Uniform flow,**

  For uniform flow, derivatives with respect to space coordinates i.e. $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, etc.$ are zero, therefore convective acceleration is zero but local acceleration is not necessarily zero.

- **For Steady & Uniform flow,**

  For steady and uniform flow, both local and convective acceleration is zero, therefore, total acceleration is zero.

3.7 Concepts of Flow Visualization

Flow visualization is useful in physical experiments, numerical solutions as well as in computational fluid dynamics (CFD). Some of the important concepts of flow visualization to describe the motion of a fluid are explained below.

3.7.1 Stream Line

"A stream line is an imaginary line drawn through a flowing fluid in such a way that, at an instant of time, the tangent at each point on the line represents the instantaneous velocity vector at that point."

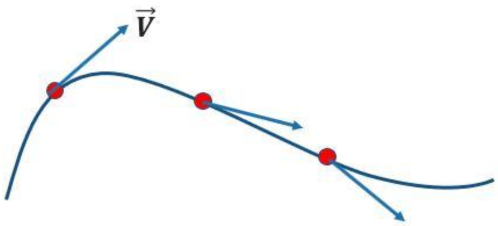

Fig.3.2 – The Stream Line

Key Points:

- There cannot be any flow across the stream line as the flow is always tangential to the stream line.
- Two stream lines cannot cross each other as otherwise there would be two velocities at that point (fluid particle at that point), which is physically impossible.
- At a different time, you will get different streamlines.
- In steady flow, the pattern of stream lines remains the same with time. So you get same streamlines at all instant of time.
- Since the velocity of the fluid particle at any point on the stream line is tangential to the stream line, there cannot be any component of velocity normal or right angle to the stream line.

Derivation of Equation of Stream Line:

Let us consider an infinitesimally small distance $d\vec{r}$ as shown in *Fig.3.3*. It becomes tangent to the stream line.
Here velocity vector,

$$\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$$

And,

$$\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$
$$\therefore d\bar{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$$

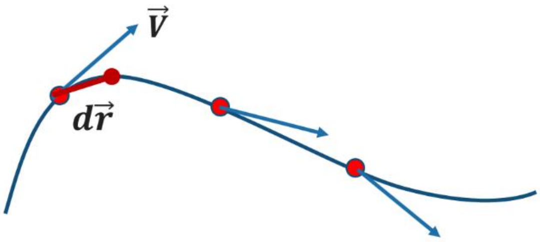

Fig.3.3 – The Stream Line

As $d\vec{r}$ and $\vec{V}$ are parallel vectors,

$$\therefore d\vec{r} \times \vec{V} = \bar{0}\ (Null\ vector)$$

$$\therefore \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ dx & dy & dz \\ u & v & w \end{bmatrix} = 0$$

Scalar equations are,

$$wdy - vdz = 0$$
$$wdx - udz = 0$$
$$vdx - udy = 0$$

Therefore, the equation of stream line is:

$$\therefore \frac{\boldsymbol{dx}}{\boldsymbol{u}} = \frac{\boldsymbol{dy}}{\boldsymbol{v}} = \frac{\boldsymbol{dz}}{\boldsymbol{w}} \qquad \textit{Eq. (3.17)}$$

3.7.2 Stream Tube

"A stream tube is an imaginary tube consists of stream lines, forming its boundary surface, which does not permit the fluid across it."

Key Points:

- Stream tube may be of regular or irregular shape.
- As stream tube is bounded by stream lines, therefore no fluid can enter or leave the stream tube from its boundary. Hence stream tube behaves as a solid surface tube.

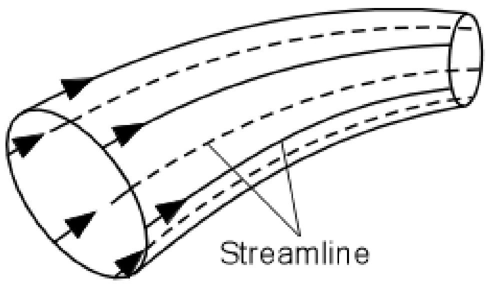

Fig.3.4 – A Stream Tube

- The general continuity equation can be applied to stream tube though it has no solid boundaries.
- In steady flow with uniform velocity, all stream lines are straight and parallel.

3.7.3 Streak Line or Filament Line

"It is an instantaneous picture of the positions of all fluid particles in the flow which have passed or emerged from a given point."

Examples:

- The line of smoke from a chimney is a streak line.
- A dye when injected into the flowing fluid and a resultant coloured lines after some time gives the streak lines.

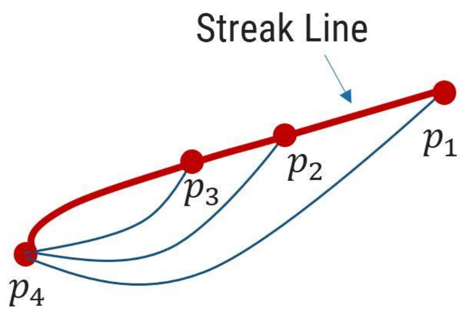

Fig.3.5 – A Streak Line

3.7.4 Path Line

"It is the actual path travelled by an individual fluid particle over a period of time."

- The locus of the same fluid particle over a time period t_1 to t_2 is called a path line.
- The path line may intersect itself at different times.
- A path line is a Langrangian concept in that we simply follow the path of an individual fluid particle as it moves around in the flow field.

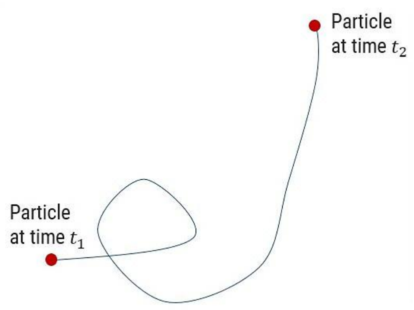

Fig.3.6 – A Path Line

3.7.5 Difference Between Stream Line, Streak Line and Path Line

*Table **3.2** - Difference between Stream line, Streak line and Path line*

Sr. No.	Stream Line	Streak Line	Path Line
1	It is an imaginary line showing positions of various fluid particles.	It is a real line showing instantaneous positions of various particles.	It is a real line showing the successive position of one particle.
2	Particles may change stream line depending on the type of flow.	May change from instant to instant.	Particles may cross its path line.
3	Stream lines cannot intersect each other, they are always parallel.	Streak line changes with time. Two streak lines may intersect each other.	Two path lines for two particles may intersect each other.
4	No flow across stream line.	Flow across the streak line is possible.	Flow across a path line is possible by other particles.

***Note:** In a steady flow, there is no geometrical distinction between the stream line, streak line and path line. They are identical if they originate at the same point.*

3.8 Rotation and Vorticity

3.8.1 Rotation (ω)

"Rotation is defined as the movement of a fluid element in such a way that both of its horizontal, as well as vertical axis, rotate in the same direction."

Flow is said to be rotational if fluid particles are rotating about their own axis while flowing along the stream line.

Rotation of a fluid particle takes place about an axis which is perpendicular to the plane formed by the two elements. E.g. in $x-y$ plane, the rotation will take place about Z-axis.

Mathematically, rotation of a fluid particle at a point is given by the average angular velocity of two perpendicular linear elements of a fluid particle.

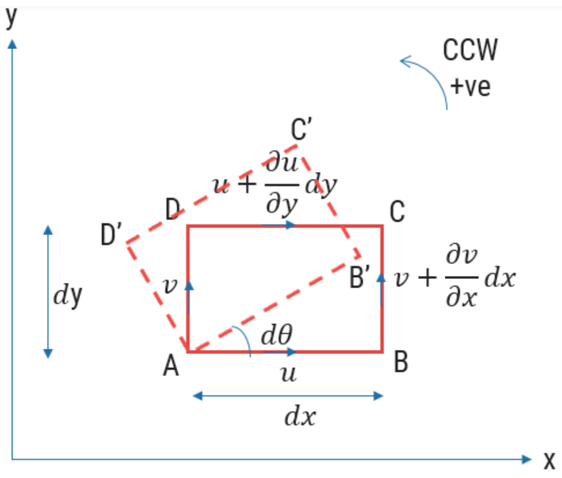

Fig.3.7 – Rotation about Z-axis

Let us consider a rectangular fluid element ABCD in $x-y$ plane as shown in *Fig.3.7*.

The velocity components in $x-$ direction at point A and D are u and $u+\frac{\partial u}{\partial y}dy$. Similarly, the velocity components in $y-$ direction at points A and B are v and $v+\frac{\partial v}{\partial x}dx$.

Since these velocities are different, there will be angular velocity developed for line AB and AD.

In time interval dt, the elements AB and AD would move relative to point A. Hence the element ABCD rotates by angle $d\theta$ and takes new position AB'C'D'.

Consider anti-clockwise rotation as positive.

Here, the distance BB' is given by,

$$Distance\ BB' = change\ in\ velocity\ \times time$$

$$\therefore Distance\ BB' = \left[v+\frac{\partial v}{\partial x}dx - v\right]\times dt$$

Similarly, distance DD' is,

$$Distance\ DD' = [u-u-\frac{\partial u}{\partial y}dy]\times dt$$

Now, the angular velocity of element AB about Z-axis is,

$$\omega_{AB} = \frac{d\theta}{dt} = \frac{(BB'/AB)}{dt}$$

$$\therefore \omega_{AB} = \frac{[v+\frac{\partial v}{\partial x}dx - v]\times dt/dx}{dt}$$

$$\therefore \omega_{AB} = \frac{\partial v}{\partial x}$$

Similarly, the angular velocity of element AD about Z-axis is,

$$\omega_{AD} = \frac{d\theta}{dt} = \frac{(DD'/AD)}{dt}$$

$$\therefore \omega_{AD} = \frac{[u-u-\frac{\partial u}{\partial y}dy]\times dt/dy}{dt}$$

$$\therefore \omega_{AD} = -\frac{\partial u}{\partial y}$$

The average of ω_{AB} and ω_{AD} will give the rotation of fluid flow about Z-axis.

$$\therefore \omega_z = \frac{1}{2}[\omega_{AB}+\omega_{AD}]$$

$$\therefore \boldsymbol{\omega_z = \frac{1}{2}\left[\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right]} \qquad \textit{Eq. (3.18)}$$

Similarly, rotation about x and y axis can be obtained as,

$$\therefore \boldsymbol{\omega_x = \frac{1}{2}\left[\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right]} \qquad \textit{Eq. (3.19)}$$

$$\therefore \boldsymbol{\omega_y = \frac{1}{2}\left[\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right]} \qquad \textit{Eq. (3.20)}$$

In vector notation,

Rotational vector,

$$\omega = \omega_x \hat{\imath} + \omega_y \hat{\jmath} + \omega_z \hat{k}$$

OR

$$\omega = \frac{1}{2}(\nabla \times v)$$

Where,

$(\nabla \times v)$ is the curl of the velocity vector.

$$(\nabla \times v) = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{bmatrix}$$

$$= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\imath} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

$$\therefore (\nabla \times v) = 2\omega_x \hat{\imath} + 2\omega_y \hat{\jmath} + 2\omega_z \hat{k}$$

$$\therefore \boldsymbol{\omega} = \frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{\nabla} \times \boldsymbol{v})$$ *Eq. (3.21)*

3.8.2 Vorticity (ξ)

- It is twice the value of rotation.

$$\therefore \xi = 2\omega = (\nabla \times v)$$

- Vorticity components can be written as,

$$\xi_x = 2\omega_x = \left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right]$$

$$\xi_y = 2\omega_y = \left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right]$$

$$\xi_z = 2\omega_z = \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$$

3.8.3 Conditions for Irrotational Flow

The irrotational flow will have both the rotation and vorticity as zero.

Irrotational flow can also be defined as:

*"When the components of rotation or vorticity are zero throughout the region in the flow then flow in that region is described as **irrotational flow**."*

- ***Therefore, for irrotational flow,***

$$\boldsymbol{\xi} = \mathbf{0}$$

$$\boldsymbol{\omega} = \mathbf{0}$$

$$(\boldsymbol{\nabla} \times \boldsymbol{v}) = \mathbf{0}$$

3.9 Circulation (Γ)

The flow along a closed curve is called circulation.

Mathematically, ***Line integral of the tangential velocity of the fluid around a closed curve is called circulation.***

Consider a closed curve or contour as shown in *Fig.3.8*.

Let, at any point on the curve the velocity of flow of fluid is v.

If α is the angle between a small element dS along the curve in tangential direction and velocity of flow v.

Then, the tangential component of velocity $= v\cos\alpha$

By definition, the circulation along a closed contour or curve is,

$$\therefore \Gamma = \int v\cos\alpha\,.\,dS$$

Here, an integral sign indicates summation around the contour.

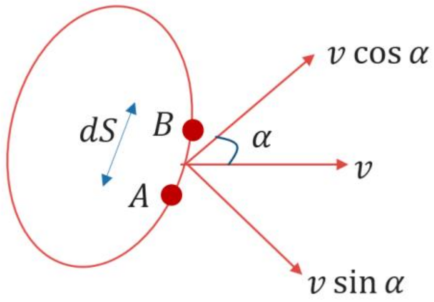

Fig.3.8 – Circulation for a closed contour

3.9.1 Circulation for Rectangular Fluid Element

In the case of rectangular fluid element ABCD, as shown in *Fig.3.9*, the total circulation is given by,

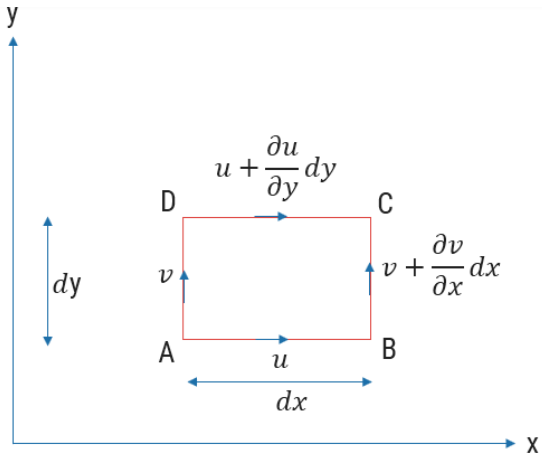

Fig.3.9 – Circulation for rectangular element

$$\Gamma = \Gamma_{AB} + \Gamma_{BC} + \Gamma_{CD} + \Gamma_{DA}$$

$$\therefore \Gamma = udx + \left[v + \frac{\partial v}{\partial x}dx\right]dy - \left[u + \frac{\partial u}{\partial y}dy\right]dx - vdy$$

$$\therefore \Gamma = \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]dxdy$$

$$\therefore \boldsymbol{\frac{\Gamma}{dxdy} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \xi_z = 2\omega_z} \qquad \text{Eq. (3.22)}$$

- Hence, vorticity can also be defined as circulation per unit area.
- For irrotational flow, vorticity=0, hence circulation around any closed path is zero.

3.10 Stream Function (Ψ)

- It is defined for a ***2-d, incompressible & steady flow*** in such a way that it will satisfy the continuity equation.
- Continuity equation for 2-D, incompressible and steady flow is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

- Here we will transform two parameters $u\ and\ v$ in terms of a single parameter, that satisfy continuity equation.

$$\therefore\therefore \boldsymbol{u = -\frac{\partial \psi}{\partial y}\ and\ v = +\frac{\partial \psi}{\partial x}} \qquad \text{Eq. (3.23)}$$

Where,

$$Stream\ Function,\ \psi = f(x, y)$$

$$\therefore d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy$$

$$\therefore d\psi = vdx - udy$$

If stream function is constant, then $d\psi = 0$

$$\therefore 0 = vdx - udy$$

$$\therefore \frac{\boldsymbol{dx}}{\boldsymbol{u}} = \frac{\boldsymbol{dy}}{\boldsymbol{v}}$$

Which is an equation of stream line. Therefore, ***stream function is always constant along a stream line.***

3.10.1 Continuity Equation for 2-D Flow in terms of Stream Function

- The continuity equation for steady, incompressible, 2-D flow is,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

By substituting the value of u and v from *Eq. (3.23)* in terms of stream function, we get,

$$\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) = 0$$

$$\therefore -\frac{\partial^2 \Psi}{\partial x \partial x} + \frac{\partial^2 \Psi}{\partial x \partial x} = 0$$

$$\therefore 0 = 0$$

- Stream function (Ψ) satisfies the continuity equation, hence the existence of Ψ means a possible case of fluid flow, which may be rotational or irrotational.

3.10.2 Laplace Equation of Stream Function for Irrotational Flow

- For 2-D flow in (x, y) plane, the rotational component ω_z is given by *Eq. (3.18)*,

$$\omega_z = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$$

By substituting the value of u and v from *Eq. (3.23)* in terms of stream function, we get,

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right)\right]$$

$$\therefore \omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right)\right]$$

$$\therefore \omega_z = \frac{1}{2}\left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right]$$

But for irrotational flow, $\omega_z = 0$

$$\therefore \frac{\boldsymbol{\partial^2 \psi}}{\boldsymbol{\partial x^2}} + \frac{\boldsymbol{\partial^2 \psi}}{\boldsymbol{\partial y^2}} = \boldsymbol{0} \qquad \textit{Eq. (3.24)}$$

Eq. (3.24) is called the Laplace equation for ψ.

3.10.3 Properties of Stream Function

The properties of the stream function are:

1) Stream function (Ψ) is constant along a stream line.
2) If stream function (Ψ) exists, it is a possible case of fluid flow, which may be rotational or irrotational.
3) If stream function (Ψ) satisfy the Laplace equation, it is a possible case of irrotational flow.

3.11 Velocity Potential Function (ϕ)

- The flow always takes place from higher to lower pressure side. This potential difference resulting in the flow of fluid is known as velocity potential.
- The velocity potential function is defined as a scalar function of space and time such that its negative derivative with respect to any direction gives the fluid velocity in that direction.

  Hence for a steady flow,

$$\phi = f(x, y, z)$$

  Then,

$$\therefore u = -\frac{\partial \phi}{\partial x}, \qquad v = -\frac{\partial \phi}{\partial y}, \qquad w = -\frac{\partial \phi}{\partial z} \qquad \textit{Eq. (3.25)}$$

  (Note: negative sign indicates that flow takes place in the direction in which velocity potential function decreases. One can consider a positive sign also.)

- For irrotational flow, we have,

$$(\nabla \times v) = 0$$

$$\left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right] = 0, \qquad \left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right] = 0, \qquad \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] = 0$$

- The velocity potential function is defined in such a way that it will satisfy these equations of irrotational flow. Hence ***if ϕ exists, the flow is irrotational.***

3.11.1 Laplace Equation of Velocity Potential Function for Irrotational Flow

- For 3-D, steady, incompressible flow, the continuity equation is,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

  By substituting the value of u, v and w from *Eq. (3.25)* in terms of velocity potential function, we get,

$$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial z}\right) = 0$$

$$\therefore \boldsymbol{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0} \qquad \textit{Eq. (3.26)}$$

  Eq. (3.26) is called the Laplace equation for ϕ.

3.11.2 Properties of Velocity Potential Function

The properties of the velocity potential function are:

1) velocity potential function (ϕ) is constant along an equipotential line.

2) If velocity potential function (ϕ) exists, the flow is irrotational (as it is defined only for irrotational flow).
3) If velocity potential function (ϕ) satisfy the Laplace equation, it represents a possible steady, incompressible and irrotational flow.

3.11.3 Equipotential Line and Stream Line are Orthogonal to Each Other

- Equipotential lines have velocity potential function as constant. For 2-D, steady, incompressible & irrotational flow,

$$\phi = f(x,y)$$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

$$\therefore d\phi = udx + vdy$$

If, $\phi = constant\ then,\ d\phi = 0$

$$\therefore 0 = udx + vdy$$

The slope of an equipotential line,

$$\therefore \frac{dy}{dx}_{\phi=c} = -\frac{u}{v}$$

- Stream function along a stream line is constant. For 2-D, steady & incompressible flow,

$$\psi = f(x,y)$$

$$\therefore d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy$$

$$\therefore d\psi = vdx - udy$$

If, $\psi = constant\ then\ d\psi = 0$

$$\therefore 0 = vdx - udy$$

The slope of streamline,

$$\therefore \frac{dy}{dx}_{\psi=c} = +\frac{v}{u}$$

- Now,

$$Slope\ of\ stream\ line \times Slope\ of\ equipotential\ line = \frac{v}{u} \times \left(-\frac{u}{v}\right) = \mathbf{-1}$$

Hence the equipotential line and stream line are orthogonal or perpendicular to each other.

3.12 Vortex Flow

Vortex flow is defined as the flow of fluid along a curved path or the flow of a rotating mass of fluid is known as Vortex flow.

The vortex is of two types:

1) Forced vortex flow
2) Free vortex flow

3.12.1 Forced Vortex Flow

Forced vortex flow is defined as a flow in which some external torque is used to rotate the fluid mass. The fluid mass in forced vortex rotates at constant angular velocity ω. Then the tangential velocity of any fluid particle is given by,

$$v = \omega \times r$$

Where r is the radius of a fluid particle from the axis of rotation.

<u>Examples of forced vortex flow:</u>

1) The flow of fluid inside the impeller of the centrifugal pump.
2) The flow of water through the runner of a turbine.
3) A liquid mass in a container rotated about its central axis with constant angular velocity.

3.12.2 Free Vortex Flow

When the fluid mass rotates without application of external torque, then it is called free vortex flow. The fluid in free vortex rotates due to the rotation imparted to the fluid previously.

<u>Examples of free vortex flow:</u>

1) A whirlpool in a river.
2) The flow of fluid through a hole provided at the bottom of a vessel.
3) The flow around a circular bend.
4) The flow in a casing of the centrifugal pump.

3.13 References

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