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## Operation Research

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## 1.1 Introduction

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The subject operations research is a branch of mathematics - especially applied mathematics used to provide a scientific base for management to take timely and effective decisions to their problems. It tries to avoid the dangers of taking decisions merely by guessing or by using thumb rules. Management is a multidimensional and dynamic concept. It is multidimensional because management problems and their solutions have consequences in several dimensions, such as human, economic social and political fields. As the manager operates his system in an environment, which will never remain static, hence is dynamic in nature. Hence any manager, while making decisions, considers all aspects in addition to the economic aspect, so that his solution should be useful in all aspects. The general approach is to analyze the problem in economic terms and then implement the solution if it does not aggressive or violent to other aspects like human, social and political constraints.

## 1.2 History of OR

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No science has ever been born on a specific day. Operations research is no exception. Its roots are as old as science and society. Though the roots of OR extend to even early 1800s, it was in 1885 when Frederick W. Taylor emphasized the application of scientific analysis to methods of production, that the real start took place.

**During World War II**, the military management in England called on a team of scientists to study the strategic and tactical problems of air and land defense. This team was under the direction of Professor P.M.S. Blackett of the University of Manchester and a former naval officer. "**Blackett's circus**", as the group was called, included *three physiologists, two mathematical physicists, one astrophysicist, one army officer, one surveyor, one general physicist, and two mathematicians*. Many of these problems were of executive-type. The objective was to find out the most effective allocation of limited military resources to the various military operations and to the activities within each operation. The application included the effective use of the newly invented radar, allocation of British Air Force Planes to missions and the determination of best patterns for searching submarines. This group of scientists formed the first OR team.

The name operations research (or operational research) was apparently coined in 1940 because the team was carrying out research on (military) operations. The encouraging results of these efforts led to the formation of more such teams in British Armed Services and the use of such scientific teams soon spread to Western Allies - the United States, Canada, and France. However this science of operations research originated in England, the United States soon took the lead. In the United States, these OR teams helped in developing strategies for mining operations, inventing new flight patterns and planning of sea mines.

**Post - World War II:** Immediately after the war, the success of military teams attracted the attention of industrial managers who were seeking solutions to their problems. Industrial operations research in the U.K. and U.S.A. developed along different lines. In the U.K. the critical economic situation required a drastic increase in production efficiency and creation of new markets. Nationalization of a few key industries further increased the potential field for OR. Consequently OR soon spread from military to government, industrial, social and economic planning.

Today, the impact of operations research can be felt in many areas. This is shown by the ever-increasing number of educational institutions offering this subject at degree level. The fast increasing number of management consulting firms speaks of the popularity of the subject. OR activities have spread to diverse fields such as hospitals, libraries, city planning, transportation systems, crime investigation, etc. Some of the Indian organizations using OR techniques are Indian - Airlines, Railways, Defence Organizations, Fertilizer Corporation of India, Delhi Cloth Mills, Tata Iron and Steel Co., etc.

## 1.3 Definitions of Operation Research

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Many definitions of OR. have been suggested from time to time. Some of the different definitions are:

1. OR. is an experimental and applied science developed to observing, understanding and predicting the behavior of purposeful man-machine systems and OR. workers are actively engaged in applying this knowledge to practical problems in business, government, and society.  
**OR. Society of America**
2. OR. is a scientific method of providing executive departments with a quantitative basis for decision regarding the operations under their control.  
**Morse and Kimbal (1946)**
3. OR. is a management activity pursued in two complementary ways – one half by the free and bold exercise of commonsense untrammelled by any routine, and the other half by the application of a repertoire of well established pre-created methods and techniques.  
**Jagjit Singh (1968)**
4. OR. is the application of scientific methods, techniques, and tools to problems involving the operations of systems so as to provide these in control of the operations with optimum solutions to the problem.  
**Churchman, Acoff, Arnoff (1957)**
5. OR. is a scientific approach to problem-solving for executive management.  
**H.M. Wagner**
6. OR. may be described as a scientific approach to decision making that involves the operations of an organizational system.  
**Hiller and Lieberman**
7. Operations Research is the art of winning wars without actually fighting.  
**Aurthur Clarke**
8. Operations Research is the art of giving bad answers to problems where otherwise worse answers are given.  
**T.L. Satty**

## 1.4 Characteristics of OR

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From the definitions of OR following characteristics can be extracted out.

### 1. Use of Interdisciplinary Teams

OR involves many numbers of variables and constraints. For a single person, it is not possible to understand and analyze justifiably. Hence people from various disciplines are required to understand the OR problem, who applies their special knowledge acquired through experience to get a better view of cause and effects of the events in the problem and to get a better solution to the problem. This type of team approach will reduce the risk of making wrong decisions.

### 2. Complete System Orientation

A business may be considered as a system having various sub-systems. The decision made by any subsystem will have its effect on other sub-systems. When dealing with OR problems, one has to consider the entire system, and characteristics of subsystems, the inter-relationship between sub-systems and then analyze the problem, search for a suitable model and get the solution for the problem. Hence it can be concluded that OR is a Systems Approach rather than an individual approach.

### 3. Involvement of Scientific Method

Various scientific methods are involved in OR to solve different kinds of problems. Scientific methods are based on derived logics and empirical relations from past experience. So the application of scientific methods leads to logical and sequential results, which are not depending on irrelevant assumptions.

### 4. Improvement in Quality of Decisions

OR provides various alternatives and let the user to select an optimal choice. This will definitely help him in making better and quick decisions. Hence, the quality of a decision can be improved.

### 5. Uncovering Hidden Problems

Sometimes, during solving the adopted problem, new problems are uncovered. These problems are mostly overlooked. For example, the excess inventory provides flexibilities in managing the orders but on other hands, it hides many problems related to manufacturing, human, finance, etc. As an uncovered problem can also affect the existing problem, it is very essential to solve these problems using different OR techniques.

## 1.5 Phases of OR

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An OR study is rooted in teamwork, where the OR analysts and the client work side by side. The OR analysts' expertise in modeling must be complemented by the experience and cooperation of the client for whom the study is being carried out.

The principal phases for implementing OR in practice include

1. Definition of the problem.
2. Construction of the model.
3. Solution of the model.
4. Validation of the model.
5. Implementation of the solution.

### 1. Definition of the problem

Problem definition involves defining the scope of the problem under investigations. This function should be carried out by the entire OR team. The aim is to identify three principal elements of the decision problem: (1) description of the decision alternatives, (2) determination of the objective of the study, and (3) specification of the limitations under which the modeled system operates.

### 2. Construction of the model

Model construction entails an attempt to translate the problem definition into mathematical relationships. If the resulting model fits one of the standard mathematical models, such as linear programming, we can usually reach a solution by using available algorithms. Alternatively, if the mathematical relationships are too complex to allow the determination of an analytic solution, the OR team may opt to simplify the model and use a heuristic approach, or they may consider the use of simulation, if appropriate. In some cases, mathematical, simulation, and heuristic models may be combined to solve the decision problem.

### 3. Solution of the model

The model solution is by far the simplest of all OR phases because it entails the use of well-defined optimization algorithms. An important aspect of the model solution phase is *sensitivity analysis*. It deals with obtaining additional information about the behavior of the optimum solution when the model undergoes some parameter changes. Sensitivity analysis is particularly needed when the parameters of the model cannot be estimated accurately. In these cases, it is important to study the behavior of the optimum solution in the neighborhood of the estimated parameters.

#### 4. Validation of the model

Model validity checks whether or not the proposed model does what it purports to do - that is, does it predict adequately the behavior of the system under study? Initially, the OR team should be convinced that the model's output does not include "surprises". In other words, does the solution make sense? Are the results intuitively acceptable? On the formal side, a common method for checking the validity of a model is to compare its output with historical output data. The model is valid if, under similar input conditions, it reasonably duplicates past performance. Generally, however, there is no assurance that future performance will continue to duplicate past behavior. Also, because the model is usually based on careful examination of past data, the proposed comparison is usually favorable.

#### 5. Implementation of the solution

The objective of OR is not only to produce reports but to improve the system performance so the results of the research must be implemented. For this, the solution obtained above should be translated into an operating procedure which can be easily understood and applied by those who control the operations. Changes needed in present procedures and resources must be clearly indicated and should be implemented. After the solution has been applied to the system, OR group must study the response of the system to the changes made. The solution obtained should, therefore, be continuously reviewed, modified and updated according to the changing situation.

### 1.6 Scope of OR

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When we broaden the scope of OR, we find that it has really been practiced for hundreds of years even before World War II. Whenever there is a problem of optimization, there is scope for the application of OR. Its techniques have been used in a wide range of situations:

#### 1. In Industry

In the field of industrial management, there is of a chain of problems starting from the purchase of raw materials to the dispatch of finished goods. The management is interested in having an overall view of the method of optimizing profits. In order to make a decision on a scientific basis, the OR team will have to consider various alternative methods of producing the goods and the return in each case. OR study should also point out the possible changes in the overall structure like the installation of a new machine, the introduction of more automation, etc. OR has been successfully applied in industry in the fields of production, blending, product mix, inventory control, demand forecast, sale and purchase, transportation, repair and maintenance, scheduling and sequencing, planning, scheduling and control of projects and scores of other associated areas.

#### 2. In Defence

OR has a wide scope for application in defense operations. In modern warfare, the defense operations are carried out by a number of different agencies, namely airforce, army, and navy. The activities performed by each of them can be further divided into sub-activities viz. operations, intelligence, administration, training and the like. There is thus a need to coordinate the various activities involved in order to arrive at optimum strategy and to achieve consistent goals. Operations research, conducted by a team of experts from all the associated fields, can be quite helpful to achieve the desired results.

### 3. Planning

In both developing and developed economies, the OR approach is equally applicable. In developing economies, there is a great scope of developing an OR approach towards planning. The basic problem is to orient the planning so that there is maximum growth of per capita income in the shortest possible time, by taking into consideration the national goals and restrictions imposed by the country. The basic problem in most of the countries in Asia and Africa is to remove poverty and hunger as quickly as possible. There is, therefore, great scope for economists, statisticians, administrators, technicians, politicians, and agriculture experts working together to solve this problem with an OR approach.

### 4. Agriculture

OR approach needs to be equally developed in the agriculture sector on a national or international basis. With the population explosion and consequent shortage of food, every country is facing the problem of optimum allocation of land to various crops in accordance with climatic conditions and available facilities. The problem of the optimal distribution of water from the various water resources is faced by each developing country and a good amount of scientific work can be done in this direction.

### 5. Public Utilities

OR methods can also be applied in big hospitals to reduce waiting time of outdoor patients and to solve the administrative problems. Monte Carlo methods can be applied in the area of transport to regulate train arrivals and their running times. Queuing theory can be applied to minimize congestion and passengers' waiting time. OR is directly applicable to business and society. For instance, it is increasingly being applied in L.I.C. office to decide the premium rates of various policies. It has also been extensively used in petroleum, paper, chemical, metal processing, aircraft, rubber, transport and distribution, mining and textile industries. OR approach is equally applicable to big and small organizations' For example, whenever a departmental store faces a problem like employing additional sales girls, purchasing an additional van, etc., techniques of OR can be applied to minimize cost and maximize the benefit for each such decision.

Thus we find that OR has a diversified and wide scope in the social, economic and industrial problems of today.

## 1.7 Limitations of OR

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### **The magnitude of Computations:**

OR tries to find out optimal solution taking into account all the factors. In modern society, these factors are enormous and expressing them in quantity and establishing relationships among these require voluminous calculations which can only be handled by machines.

### **Non-Quantifiable Factors:**

OR provides a solution only when all elements related to a problem can be quantified. All relevant variables do not lend themselves to quantification. Factors which cannot be quantified, find no place in OR Models in OR do not take into account qualitative factors or economical factors which may be quite important.

### **Distance between Manager and Operations Research:**

OR being a specialist's job requires a mathematician or a statistician, who might not be aware of the business problems. Similarly, a manager fails to understand the complex working of OR. Thus there is a gap between the two. Management itself may offer a lot of resistance due to conventional thinking.

## Money and Time Costs:

When the basic data are subjected to frequent changes, incorporating them into the OR models is a costly affair. Moreover, a fairly good solution at present may be more desirable than a perfect OR solution available after sometimes.

## Implementation:

Implementation of decisions is a delicate task. It must take into account the complexities of human relations and behavior. Sometimes resistance is offered only due to psychological factors.

## 1.8 Operations Research and Decision Making

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Operations research or management science, as the name suggests, is the science of managing. As is known, management is most of the time making decisions. It is thus a decision science which helps management to make better decisions. The decision is, in fact, a pivotal word in managing. It is not only the headache of management; rather all of us make decisions.

Decision-making can be improved and, in fact, there is a scope of large scale improvement. The essential characteristics of all decisions are

- a) Objectives,
- b) Alternatives,
- c) Influencing factors (constraints).

Once these characteristics are known, one can think of improving the characteristics so as to improve upon the decisions itself.

For, example, if you have to reach at an exam centre within time, using various modes of conveyance. You have three conveyance modes. First is to go alone in own car, second is to go by city bus and the third one is to go in sharing car among friends. If you choose the first conveyance mode than you will definitely reach the exam centre on time, but it will be definitely costly. If you choose the second conveyance mode, it will be least costly, but it might take more time in traveling, so you might get late at the exam centre. If you choose to travel by sharing among friends, it will let you reach within time, as well as it will be both economical compared to the first options. Although the destination was reached by using any conveyance mode, the best option is to select the third alternate, which is economical and object-oriented.

It is scientific quantification used in OR, which helps management to make better decisions. Thus in OR, the essential features of decisions, namely, objectives, alternatives and influencing factors are expressed in terms of scientific quantifications or mathematical equations. This gives rise to certain mathematical relations, termed a whole as a mathematical model. Thus the essence of OR is such a mathematical model.

However, with the advance of science and technology, decision-making in business and industry has become highly complex and extremely difficult. The decision-maker is not only faced with a large number of interacting variables, which at times do not lend themselves to neat quantitative treatment but also finds them too numerous and dynamic. Above all he has to take into consideration the action of the competitors over which he has no control. This complexity of decision-making make made the decision-makers look for various aids in decision-making. It is in these situations that operations research comes to our help.

The managers today make full use of the OR techniques in various functional areas. It has been realized beyond doubt that intuition alone has no place in decision-making since such a decision becomes highly questionable when it involves the choice among several alternatives. OR provides the management with much-needed tools for improving the various decisions.

## 1.9 Introduction to Linear Programming Problem

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- ▶ Many business and economic activities are concerned with the problem of planning. If the supply of the resources are unlimited, the need for linear programming problems would not arise at all. In each case, limited resources are available, and to make use of this resource, programming and planning problems could be formulated to maximise/minimise a linear form of profit/cost function whose variables are restricted to values satisfying a system of linear constraints (a set of linear equations/or inequations). The term 'programming' refers to the process of determining a plan of action. Linear programming is a technique for determining an optimum schedule of independent activities in view of the available resources. Linear relationship between two or more variables is the one in which the variables are directly or precisely proportional.
- ▶ Linear programming is perhaps the most widely applied mathematical technique that helps managers in decision-making and planning for the optimal allocation of limited resources. It deals with the optimization (maximization or minimization) of a function of variables known as *objective function*, subject to a set of linear equations and/or inequalities known as *constraints*. The objective function may be profit, cost, production capacity or any other measure of effectiveness, which is to be obtained in the best possible or optimal manner. The constraints may be imposed by different resources such as raw material availability, market demand, production process and equipment, storage capacity, etc. By linearity is meant a mathematical expression in which the expressions among the variables are linear e.g., the expression  $a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n$  is linear. Higher powers of the variables or their products donot appear in the expressions for the objective function as well as the constraints (they donot have expressions like  $x_1^3$ ,  $x_2^{3/2}$ ,  $x_1x_2$ ,  $a_1x_1 + a_2 \log x_2$ , etc.). The variables obey the properties of *proportionality* (e.g., if a product requires 3 hours of machining time, 5 units of it will require 15 hours) and *additivity* (e.g., amount of a resource required for a certain number of products is equal to the sum of the resource required for each).
- ▶ It was in 1947 that George Dantzig and his associates found out a technique for solving military planning problems while they were working on a project for U.S. Air Force. This technique consisted of representing the various activities of an organization as a linear programming (L.P.) model and arriving at the optimal programme by minimizing a linear objective function. Afterwards, Dantzig suggested this approach for solving business and industrial problems. He also developed the most powerful mathematical tool known as "simplex method" to solve linear programming problems.

## 1.10 Requirements for A Linear Programming Problem

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- ▶ All organizations, big or small, have at their disposal, men, machines, money and materials, the supply of which may be limited. If the supply of these resources were unlimited, the need for management tools like linear programming would not arise at all. Supply of resources being limited, the management must find the best allocation of its resources in order to maximize the profit or minimize the cost/loss or utilize the production capacity to the maximum extent. However, this involves a number of problems which can be overcome by quantitative methods, particularly the linear programming.
- ▶ Generally speaking, linear programming can be used for optimization problems if the following conditions are satisfied:
  1. There must be a well defined objective function (profit, cost or quantities produced) which is to be either maximized or minimized and which can be expressed as a linear function of decision variables.
  2. There must be constraints on the amount or extent of attainment of the objective and these constraints must be capable of being expressed as linear equations or inequalities in terms of variables.



3. There must be alternative courses of action. For example, a given product may be processed by two different machines and problem may be as to how much of the product to allocate to which machine.
4. Another necessary requirement is that decision variables should be interrelated and nonnegative. The non-negativity condition shows that linear programming deals with real life situations for which negative quantities are generally illogical.
5. As stated earlier, the resources must be in limited supply. For example, if a firm starts producing greater number of a particular product, it must make smaller number of other products as the total production capacity is limited.

## 1.11 Assumptions In Linear Programming Models

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- ▶ A linear programming model is based on the following assumptions:

### 1. Proportionality

A basic assumption of linear programming is that proportionality exists in the objective function and the constraints. This assumption implies that if a product yields a profit of ₹ 100, the profit earned from the sale of 15 such products will be ₹ (100 x 15) = ₹ 1500. This may not always be true because of quantity discounts. Further, even if the sale price is constant, the manufacturing cost may vary with the number of units produced and so may vary the profit per unit. Likewise, it is assumed that if one product requires processing time of 2 hours, then ten such products will require processing time of 2 x 10 = 20 hours. This may also not be true as the processing time per unit often decreases with increase in number of units produced. The real world situations may not be strictly linear. However, assumed linearity represents their close approximations and provides very useful answers.

### 2. Additivity

It means that if we use  $t_1$  hours on machine A to make product 1 and  $t_2$  hours to make product 2, the total time required to make products 1 and 2 on machine A is  $t_1 + t_2$  hours. This, however, is true only if the change-over time from product 1 to product 2 is negligible. Some processes may not behave in this way. For example, when several liquids of different chemical compositions are mixed, the resulting volume may not be equal to the sum of the volumes of the individual liquids.

3.

### 3. Continuity

Another assumption underlying the linear programming model is that the decision variables are continuous i.e., they are permitted to take any non-negative values that satisfy the constraints. However, there are problems wherein variables are restricted to have integral values only. Though such problems, strictly speaking, are not linear programming problems, they are frequently solved by linear programming techniques and the values are then rounded off to nearest integers to satisfy the constraints. This approximation, however, is valid only if the variables have large optimal values. Further, it must be ascertained whether the solution represented by the rounded values is a feasible solution and also whether the solution is the best integer solution.

### 4. Certainty

Another assumption underlying a linear programming model is that the various parameters, namely, the objective function coefficients, R.H.S. coefficients of the constraints and resource values in the constraints are certainly and precisely known and that their values do not change with time. Thus the profit or cost per unit of the product, labour and materials required per unit, availability of labour and materials, market demand of the product produced, etc. are assumed to be known with certainty. The linear programming problem is, therefore, assumed to be deterministic in nature.

### 5. Finite Choices

A linear programming model also assumes that a finite (limited) number of choices (alternatives) are available to the decision-maker and that the decision variables are interrelated and non negative. The non-negativity condition shows that linear programming deals with real-life situations as it is not possible to produce/use negative quantities. Mathematically these non-negativity conditions do not differ from other constraints. However, since while solving the problems they are handled differently from the other constraints, they are termed as non-negativity restrictions and the term constraints is used to represent constraints other than non-negativity restrictions.

## 1.12 Applications of Linear Programming Method

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- ▶ Though, in the world we live, most of the events are non-linear, yet there are many instances of linear events that occur in day-to-day life. Therefore, an understanding of linear programming and its application in solving problems is utmost essential for today's managers.
- ▶ Linear programming techniques are widely used to solve a number of business, industrial, military, economic, marketing, distribution and advertising problems. Three primary reasons for its wide use are:
  1. A large number of problems from different fields can be represented or at least approximated to linear programming problems.
  2. Powerful and efficient techniques for solving L.P. problems are available.
  3. L.P models can handle data variation (sensitivity analysis) easily.
- ▶ However, solution procedures are generally iterative and even medium size problems require manipulation of large amount of data. But with the development of digital computers, this disadvantage has been completely overcome as these computers can handle even large L.P. problems in comparatively very little time at a low cost.

## 1.13 Areas of Application Of Linear Programming

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- ▶ Linear programming is one of the most widely applied techniques of operations research in business, industry and numerous other fields. A few areas of its application are given below.

### 1.13.1 Industrial Applications

#### a) Product mix problems

An industrial organisation has available a certain production capacity (men, machines, money, materials, market, etc.) on various manufacturing processes to manufacture various products. Typically, different products will have different selling prices, will require different amounts of production capacity at the several processes and will therefore, have different unit profits; there may also be stipulations (conditions) on maximum and/or minimum product levels. The problem is to determine the product mix that will maximize the total profit.

#### b) Blending problems

These problems are likely to arise when a product can be made from a variety of available raw materials of various compositions and prices. The manufacturing process involves blending (mixing) some of these materials in varying quantities to make a product of the desired specifications.

For instance, different grades of gasoline are required for aviation purposes. Prices and specifications such as octane ratings, tetra ethyl lead concentrations, maximum vapour pressure, etc. of input ingredients are given and the problem is to decide the proportions of these ingredients to make the desired grades of gasoline so that ( / ) maximum output is obtained and ( // ) storage capacity restrictions are satisfied. Many similar situations such as preparation of different kinds

of whisky, chemicals, fertilisers and alloys, etc. have been handled by this technique of linear programming.

**c) Production scheduling problems**

They involve the determination of optimum production schedule to meet fluctuating demand. The objective is to meet demand, keeping inventory and employment at reasonable minimum levels, while minimizing the total cost of production and inventory.

**d) Trim loss problems**

They are applicable to paper, sheet metal and glass manufacturing industries where items of standard sizes have to be cut to smaller sizes as per customer requirements with the objective of minimizing the waste produced.

**e) Assembly-line balancing**

It relates to a category of problems wherein the final product has a number of different components assembled together. These components are to be assembled in a specific sequence or set of sequences. Each assembly operator is to be assigned the task / combination of tasks so that his task time is less than or equal to the cycle time.

**f) Make-or-buy (sub-contracting) problems**

They arise in an organisation in the face of production capacity limitations and sudden spurt in demand of its products. The manufacturer, not being sure of the demand pattern, is usually reluctant to add additional capacity and has to make a decision regarding the products to be manufactured with his own resources and the products to be sub-contracted so that the total cost is minimized.

### 1.13.2 Management Applications

**a) Media selection problems**

They involve the selection of advertising mix among different advertising media such as T.V., radio, magazines and newspapers that will maximize public exposure to company's product. The constraints may be on the total advertising budget, maximum expenditure in each media, maximum number of insertions in each media and the like.

**b) Portfolio selection problems**

They are frequently encountered by banks, financial companies, insurance companies, investment services, etc. A given amount is to be allocated among several investment alternatives such as bonds, saving certificates, common stock, mutual fund, real estate, etc. to maximize the expected return or minimize the expected risk.

**c) Profit planning problems**

They involve planning profits on fiscal year basis to maximize profit margin from investment in plant facilities, machinery, inventory and cash on hand.

**d) Transportation problems**

They involve transportation of products from, say,  $n$  sources situated at different locations to, say,  $m$  different destinations. Supply position at the sources, demand at destinations, freight charges and storage costs, etc. are known and the problem is to design the optimum transportation plan that minimizes the total transportation cost (or distance or time).

**e) Assignment problems**

They are concerned with allocation of facilities (men or machines) to jobs. Time required by each facility to perform each job is given and the problem is to find the optimum allocation (one job to one facility) so that the total time to perform the jobs is minimized.

**f) Manpower scheduling problems**

They are faced by big hospitals, restaurants and companies operating in a number of shifts. The problem is to allocate optimum man-power in each shift so that the overtime cost is minimized.

### 1.13.3 Miscellaneous Applications

#### a) Diet problems

They form another important category to which linear programming has been applied. Nutrient contents such as vitamins, proteins, fats, carbohydrates, starch, etc. in each of a number of food stuffs is known. Also the minimum daily requirement of each nutrient in the diet as well as the cost of each type of food stuff is given and the problem is to determine the minimum cost diet that satisfies the minimum daily requirement of nutrients.

#### b) Agriculture problems

These problems are concerned with the allocation of input resources such as acreage of land, water, labour, fertilisers and capital to various crops so as to maximize net revenue.

#### c) Flight scheduling problems

They are devoted to the determination of the most economical patterns and timings of flights that result in the most efficient use of aircrafts and crews.

#### d) Environment protection

They involve analysis of different alternatives for efficient waste disposal, paper recycling and energy policies.

#### e) Facilities location

These problems are concerned with the determination of best location of public parks, libraries and recreation areas, hospital ambulance depots, telephone exchanges, nuclear power plants, etc.

### 1.14 Mathematical Formulation of Linear Programming Problem

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- ▶ The procedure for mathematical formulation of LPP consists of the following steps:
  - **Step 1:** Identify the decision variables of the problem.
  - **Step 2:** Formulate the objective function to be optimised (maximised or minimised) as a linear function of the decision variables.
  - **Step 3:** Formulate the constraints of the problem such as resource limitations, market conditions, interrelation between variables and others as linear equation or inequations in terms of the decision variables.
  - **Step 4:** Add the non-negativity constraint so that negative values of the decisions variables do not have any valid physical interpretation.
- ▶ The objective function, the set of constraint and the non-negative constraint together form a linear programming problem.

### 1.15 Graphical Method of Solution

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- ▶ Once a problem is formulated as mathematical model, the next step is to solve the problem to get the optimal solution. A linear programming problem with only two variables presents a simple case, for which the solution can be derived using a graphical or geometrical method. Though, in actual practice such small problems are rarely encountered, the graphical method provides a pictorial representation of the solution process and a great deal of insight into the basic concepts used in solving large L.R problems. This method consists of the following steps:
  1. Represent the given problem in mathematical form i.e., formulate the mathematical model for the given problem.
  2. Draw the  $x_1$  and  $x_2$  axis. The non-negativity restrictions  $x_1 \geq 0$  and  $x_2 \geq 0$  imply that the values of the variables  $x_1$  and  $x_2$  can lie only in the first quadrant. This eliminates a number of infeasible alternatives that lie in 2<sup>nd</sup>, 3<sup>rd</sup> and 4<sup>th</sup> quadrants.

3. Plot each of the constraint on the graph. The constraints, whether equations or inequalities are plotted as equations. For each constraint, assign any arbitrary value to one variable and get the value of the other variable. Similarly, assign another arbitrary value to the other variable and find the value of the first variable. Plot these two points and connect them by a straight line. Thus each constraint is plotted as line in the first quadrant.
4. Identify the feasible region (or solution space) that satisfies all the constraints simultaneously. For  $\geq$  type constraint, the area on or above the constraint line i.e., away from the origin and for  $\leq$  type constraint, the area on or below the constraint line i.e., towards origin will be considered. The area common to all the constraints is called feasible region and is shown shaded. Any point on or within the shaded region represents a feasible solution to the given problem. Though a number of infeasible points are eliminated, the feasible region still contains a large number of feasible points. Feasible region is also called convex polygen. Which is a convex set formed by the intersection of finite number of closed half-planes.
5. Use iso-profit (cost) function line approach. For this, plot the objective function by assuming  $Z = 0$ . This will be a line passing through the origin. As the value of  $Z$  is increased from zero, the line starts moving to the right, parallel to itself. Draw lines parallel to this line till the line is farthest way from the origin (for a maximization problem). For a minimization problem, the line will be nearest to the origin. The point of the feasible region through which this line passes will be the optimal point. It is possible that this line may coincide with one of the edges of the feasible region. In that case, every point on that edge will give the same maximum/minimum value of the objective function and will be the optimal point.

*Alternatively use extrema point enumeration approach.* For this, find the co-ordinates of each extreme point (or corner point or vertex) of the feasible region. Find the value of the objective function at each extreme point. The point at which objective function is maximum/minimum is the optimal point and its co-ordinates give the optimal solution.

## 1.16 The General Linear Programming Problem

- ▶ The general linear programming problem can be expressed as follows:
- ▶ Find the values of variables  $x_1, x_2, \dots, x_n$  which maximize (or minimize) an objective function which is a linear function of variables, such as

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n \quad \text{Eq. (1.1)}$$

subject to the constraints

$$\left. \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n (\leq, =, \geq) b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n (\leq, =, \geq) b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n (\leq, =, \geq) b_m \end{array} \right\} \quad \text{Eq. (1.2)}$$

and meet the non-negativity restrictions

$$x_1, x_2, \dots, x_n \geq 0 \quad \text{Eq. (1.3)}$$

- ▶ For each constraint one and only one of signs ( $\leq, =, \geq$ ) holds but the sign may vary from one constraint to another.

## 1.17 Theory of Simplex Method

- ▶ Simplex method, also called simplex technique or simplex algorithm was developed in 1947 by G.B. Dantzig, an American mathematician. It has the advantage of being universal, i.e., any linear model for which the solution exists, can be solved by it. In principle, it consists of starting with a certain solution

of which all that we know is that it is basic feasible, i.e., it satisfies the constraints as well as non-negativity conditions ( $x_j \geq 0, j = 1, 2, 3, \dots, n$ ). We, then, improve upon this solution at consecutive stages, until, after a certain finite number of stages, we arrive at the optimal solution. The method also helps the decision-maker to identify the redundant constraints, an unbounded solution, multiple solutions and an infeasible solution.

- ▶ The simplex method provides an algorithm which consists in moving from one vertex of the region of feasible solutions to another in such a manner that the value of the objective function at the succeeding vertex is less in a minimization problem (or more in a maximization problem) than at the preceding vertex. This procedure of jumping from one vertex to another is then repeated. Since the number of vertices is finite, this method leads to an optimal vertex in a finite number of steps. The basis of the simplex method consists of two fundamental conditions:

1. **The feasibility condition:** It ensures that if the starting solution is basic feasible, only basic feasible solutions will be obtained during computation.
2. **The optimality condition:** It guarantees that only better solutions (as compared to the current solution) will be encountered.

## 1.18 Some Important Definitions

- ▶ Consider the general linear programming problem involving  $n$  variables and  $m$  constraints ( $m \leq n$ ):
- ▶ Determine the values of variables  $x_1, x_2, \dots, x_n$  which

maximize

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

subject to

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1,$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2,$$

$$\vdots \qquad \qquad \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m,$$

where  $x_1, x_2, \dots, x_n \geq 0$ .

- ▶ Introducing slack variables  $s_1, s_2, \dots, s_n$  in the constraints, it can be put in the following standard form :

maximize

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n \qquad \qquad \qquad \text{Eq. (1.4)}$$

subject to

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + s_1 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + s_2 = b_2$$

$$\vdots \qquad \qquad \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + s_m = b_m$$

Eq. (1.5)

where

$$x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_m \geq 0 \qquad \qquad \qquad \text{Eq. (1.6)}$$

### 1. Solution

A set of variables  $[x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_m]$  is called a solution to L.P. problem if it satisfies the constraints (2.5).

### 2. Feasible solution

A set of variables  $[x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_m]$  is called a feasible solution to L.P. problem if it satisfies the constraints (2.5) as well as non-negativity restrictions (2.6).

### 3. Basic solution

A solution obtained by setting any  $n$  variables (among  $m + n$  variables) equal to zero and solving for remaining  $m$  variables (provided the determinant of the coefficients of these  $m$  variables is non-zero)

is called a basic solution. These  $m$  variables (some of them may be zero) are called basic variables and the remaining  $n$  variables that have been put equal to zero each are called non-basic variables.

#### 4. Basic feasible solution

It is a basic solution that also satisfies the non-negativity restrictions (2.6). All variables in a basic feasible solution are  $\geq 0$ . Every basic feasible solution of a problem is an extreme point of the convex set of feasible solutions and every extreme point is a basic feasible solution of the set of constraints.

#### 5. Non-degenerate basic feasible solution

It is a basic feasible solution in which all the  $m$  basic variables are positive ( $> 0$ ) and the remaining  $n$  variables are zero each.

#### 6. Degenerate basic feasible solution

It is a basic feasible solution in which one or more of the  $m$  basic variables are equal to zero.

#### 7. Optimal basic feasible solution

It is the basic feasible solution that also optimizes the objective function (2.4).

#### 8. Unbounded solution

If the value of the objective function can be increased or decreased indefinitely, the solution is called unbounded solution.

### 1.19 The Simplex Method / Technique / Algorithm

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- ▶ The graphical method cannot be applied when the number of variables involved in the L.P. problem is more than three or rather two, since even with three variables the graphical solution becomes tedious as it involves intersection of planes in three dimensions. The simplex method, developed by Prof. George B. Dantzig, can be used to solve any L.P. problem (for which the solution exists) involving any number of variables and constraints (hundreds or even thousands).
- ▶ The computational procedure in the simplex method is based on the fundamental property that the optimal solution to an L.P. problem, if it exists, occurs only at one of the corner points of the feasible region. The simplex method always starts with initial basic feasible solution i.e., origin, which is one of the corner points of the feasible region. This solution is then tested i.e., it is ascertained whether improvement in the value of the objective function is possible by moving to the next corner point of the feasible region. If so, the solution at this point is obtained. This search for better corner point is repeated, till after a finite number of trials, the optimal solution, if it exists, is obtained.

### 1.20 Artificial Variables Techniques

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- ▶ The problems in which constraints were of ( $\leq$ ) type (with non-negative right-hand sides). The introduction of slack variables readily provided the initial basic feasible solution. There are, however, many linear programming problems where slack variables cannot provide such a solution. In these problems at least one of the constraints is of ( $\geq$ ) or ( $=$ ) type.
- ▶ In such cases, we introduce another type of variables called artificial variables. These variables are fictitious and have no physical meaning. They assume the role of slack variables in the first iteration, only to be replaced at a later iteration. Thus they are merely a device to get the starting basic feasible solution so that simplex algorithm can be applied as usual to get optimal solution.
- ▶ There are two (closely related) techniques available to solve such problems. They are
  1. **The big M-method** or M-technique or method of penalties due to A. Charnes.
  2. **The two-phase method** due to Dantzig, Orden and Wolfe.

#### 1.20.1 The Big M-Method

- ▶ This method consists of the following basic steps :

- ▶ **Step 1.** Express the linear programming problem in standard form by introducing slack variables. These variables are added to the left-hand sides of the constraints of ( $\leq$ ) type and subtracted from the constraints of ( $\geq$ ) type.
- ▶ **Step 2.** Add non-negative variables to the left-hand sides of all the constraints of initially ( $\geq$ ) or ( $=$ ) type. These variables are called artificial variables. The purpose of introducing the artificial variables is just to obtain an initial basic feasible solution. They have, however, two drawbacks:
  1. They are fictitious, have no physical meaning or economic significance and have no relevance to the problem.
  2. Their introduction (addition) violates the equality of constraints that has been already established in step 1.
- ▶ They are, therefore, rightly termed as artificial variables as opposed to other real decision variables in the problem. Therefore, we must get rid of these variables and must not allow them to appear in the final solution. To achieve this, these variables are assigned a very large per unit penalty in the objective function. This penalty is designated by  $-M$  for maximization problems and  $+M$  for minimization problems, where  $M > 0$ . Value of  $M$  is much higher than the cost coefficients of other variables and for hand calculations it is not necessary to assign any specific value to it.
- ▶ **Step 3.** Solve the modified linear programming problem by the simplex method.
- ▶ While making iterations, using the simplex method, one of the following three cases may arise:
  1. If no artificial variable remains in the basis and the optimality condition is satisfied, then the solution is an optimal feasible solution to the given problem. Also, the original constraints are consistent and none of them is redundant.
  2. If at least one artificial variable appears in the basis at zero level (with zero value in 6-column) and the optimality condition is satisfied, then the solution is optimal feasible (though degenerate) solution to the given problem. The constraints are consistent though redundancy may exist in them. By redundancy is meant that the problem has more than the required number of constraints.
  3. If at least one artificial variable appears in the basis at a non-zero level (with positive value in 6-column) and the optimality condition is satisfied, then the original problem has no feasible solution; for if a feasible solution existed, the artificial variables could be driven to zero, yielding an improved value of the objective function. The problem has no feasible solution either because the constraints are inconsistent or because there are solutions, but none is feasible. In economic terms this means that the resources of the system are not sufficient to meet the expected demands. The final solution to the problem is not optimal since the objective function contains an unknown quantity  $M$ . Such a solution satisfies the constraints but does not optimize the objective function and is also called pseudo-optimal solution.

### 1.20.2 The Two-Phase Method

- ▶ In the preceding section we observed that it was frequently necessary to add artificial variables to the constraints to obtain an initial basic feasible solution to an L.P. problem. If the problem is to be solved, the artificial variables must be driven to zero. The two-phase method is another method to handle these artificial variables. Here the L.P. problem is solved in two phases.

#### Phase I

- ▶ In this phase we find an i.b.fs. to the original problem. For this all artificial variables are to be driven to zero. To do this an artificial objective function ( $w$ ) is created which is the sum of all the artificial variables. This new objective function is then minimized, subject to the constraints of the given (original) problem, using the simplex method. At the end of phase I, three cases arise:



1. If the minimum value of  $w > 0$ , and at least one artificial variable appears in the basis at a positive level, then the given problem has no feasible solution and the procedure terminates.
2. If the minimum value of  $w = 0$ , and no artificial variable appears in the basis, then a basic feasible solution to the given problem is obtained. The artificial variable column (5) is/are deleted for phase II computations.
3. If the minimum value of  $w = 0$  and one or more artificial variables appear in the basis at zero level, then a feasible solution to the original problem is obtained. However, we must take care of this artificial variable and see that it never becomes positive during phase II computations. Zero cost coefficient is assigned to this artificial variable and it is retained in the initial table of phase II. If this variable remains in the basis at zero level in all phase II computations, there is no problem. However, the problem arises if it becomes positive in some iteration. In such a case, a slightly different approach is adopted in selecting the outgoing variable. The lowest non-negative replacement ratio criterion is not adopted to find the outgoing variable. Artificial variable (or one of the artificial variables if there are more than one) is selected as the outgoing variable. The simplex method can then be applied as usual to obtain the optimal basic feasible solution to the given L.P. problem.

## Phase II

- ▶ When phase I results in (2) or (3), we go on to phase II to find optimum solution to the given L.P. problem. The basic feasible solution found at the end of phase I is now used as a starting solution for the original problem. In other words, the final table of phase I becomes the starting table of phase II in which the artificial (auxiliary) objective function is replaced by the original objective function. The simplex method is then applied to arrive at the optimum solution. Artificial variables which do not appear in the basis may be deleted.

## 1.21 Sensitivity Analysis

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- ▶ Once the optimal solution to a linear programming problem has been attained, two situations may arise which require additional computations
  - a. During the formulation it is assumed that the parameters such as market demand, equipment capacity, resource consumption, resource availability, the relevant costs or profits are all known with certainty and do not change over time. In actual practice the markets fluctuate, material and labour costs go up or down, production times change and equipment availability varies from time to time. It is, therefore, desirable to study how the current optimal solution changes when the parameters of the problem get changed. In these problems this information may be more important than the single result provided by the optimal solution. Such an analysis converts the static linear programming solution into a dynamic tool to study the effect of changing conditions such as in business and industry.
  - b. The second situation is rather unpleasant, yet one may be encountered with it quite often. After attaining the optimal solution, one may discover that a wrong value of a cost coefficient was used or a particular variable or constraint was omitted or one or more of right-hand constants used were wrong.
- ▶ The changes in parameters of the problem may be discrete or continuous. The study of the effect of discrete changes in parameters on the optimal solution is called the sensitivity analysis or the post optimality analysis, while that of continuous changes in parameters is called parametric programming. One way to determine the effects of parameter changes is to solve the problem anew, which may be computationally inefficient. Alternatively, the current optimal solution may be investigated, making use of the properties of the simplex criterion. The second method reduces additional computations considerably and hence forms the subject of the present discussion.

The changes in the parameters of a linear programming problem include:

1. Changes in the right-hand side of the constraints or availability of resources ( $b_i$ ).
2. Changes in the cost / profit coefficients or cost/profit contribution per unit of decision variables ( $c_j$ ).
3. Addition of new variables.
4. Changes in the coefficients of constraints or consumption of resources per unit of decision variables ( $a_{ij}$ )
5. Addition of new constraints.
6. Deletion of variables.
7. Deletion of constraints.

Generally, these parameter changes result in one of the following three cases:

1. The optimal solution remains unchanged i.e., the basic variables and their values remain unchanged.
  2. The basic variables remain unchanged but their values change.
  3. The basic variables as well as their values are changed.
- ▶ While dealing with these changes, one important objective is to find the maximum extent to which a parameter or a set of parameters can be changed so that the current optimal solution remains optimal. In other words, the objective is to determine how sensitive is the optimal solution to the changes in those parameters. Such an analysis is called sensitivity analysis.

## 1.22 Duality in Linear Programming

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- ▶ For every L.P. problem (linear programme) there is a related unique L.P. problem (another linear programme) involving the same data which also describes the original problem (programme).
- ▶ The given original programme is called the primal programme (P). This programme can be rewritten by transposing (reversing) the rows and columns of the algebraic statement of the problem. Inverting the programme in this way results in dual programme (D). The variables of the dual programme (problem) are known as dual variables or shadow prices of the various resources. A solution to the dual programme may be found in a manner similar to that used for the primal. The two programmes have very closely related properties so that optimal solution of the dual problem gives complete information about the optimal solution of the primal problem and vice versa.
- ▶ Duality is an extremely important and interesting feature of linear programming. The various useful aspects of this property are
- a. If the primal problem contains a large number of rows (constraints) and a smaller number of columns (variables), the computational procedure can be considerably reduced by converting it into dual and then solving it. Hence it offers an advantage in many applications.
  - b. It gives additional information as to how the optimal solution changes as a result of the changes in the coefficients and the formulation of the problem. This forms the basis of post optimality or sensitivity analysis.
  - c. Duality in linear programming has certain far reaching consequences of economic nature. This can help managers to compare the alternative courses of action and their relative values.
  - d. Calculation of the dual checks the accuracy of the primal solution.
  - e. Duality in linear programming shows that each linear programme is equivalent to a two-person zero-sum game. This indicates that fairly close relationships exist between linear programming and the theory of games.
  - f. Duality is not restricted to linear programming problems only but finds application in economics, physics and other fields. In economics it is used in the formulation of input and output systems. In physics it is used in the series circuit and parallel circuit theory.
  - g. Economic interpretation of the dual helps the management in making future decisions.

