1

Dynamics Force Analysis of Mechanisms

Contents

1.1 Introduction

Dynamic forces are associated with accelerating masses. As all machines have some accelerating parts, dynamic forces are always present when the machines operate. In situations where dynamic forces are dominant or comparable with magnitudes of external forces and operating speeds are high, dynamic analysis has to be carried out. For example, in case of rotors which rotate at speeds more than 80000 rpm, even the slightest eccentricity of the centre of mass from the axis of rotation produces very high dynamic forces. This may lead to vibrations, wear, noise or even machine failure.

1.2 D'Alembert's Principle

D'Alembert's principle states that the inertia forces and couples, and the external forces and torques on a body together give statical equilibrium.

Inertia is a property of matter by virtue of which a body resists any change in velocity.

$$\text{Inertia force } F_i = -m\, f_g$$

Where m = mass of body

f_g = acceleration of centre of mass of the body

The negative sign indicates that the force acts in the opposite direction to that of the acceleration. The force acts through the centre of mass of the body.

Similarly, an inertia couple resists any change in the angular velocity.

$$\text{Inertia couple, } C_i = -\, I_g \alpha$$

Where I_g = moment of inertia about an axis passing through the centre of mass G and perpendicular to plane of rotation of the body

α = angular acceleration of the body

Let $\Sigma F = F_1, F_2, F_3$, etc. = external forces on the body

and $\Sigma T = T_{g1}, T_{g2}, T_{g3}$, etc. = external torques on the body about the centre of mass G.

According to D' Alembert's principle, the vector sum of forces and torques (or couples) has to be zero, i.e.,

$$\Sigma F + F_i = 0$$

$$\Sigma T + C_i = 0$$

These equations are similar to the equation o f a body in static equilibrium, i.e., $\Sigma F = 0$ and $\Sigma T = 0$.

This suggests that first the magnitudes and the directions of inertia forces and couples can be determined, after which they can be treated just like static loads on the mechanism. Thus, a dynamic analysis problem is reduced to one requiring static analysis.

1.3 Equivalent Offset Inertia Force

In plane motions involving accelerations, the inertia force acts on a body through its centre of mass. However, if the body is acted upon by forces such that their resultant does not pass through the centre of mass, a couple also acts on the body. In graphical solutions, it is possible to replace inertia force and inertia couple by an equivalent offset inertia force which can account for both. This is done by displacing the line of action of the inertia force from the centre of mass. The perpendicular displacement h of the force from the centre of mass is such that the torque so produced is equal to the inertia couple acting on the body,

$$T_i = C_i$$

$$F_i \times h = C_i$$

$$h = \frac{C_i}{F_i} = \frac{-I_g\alpha}{-m f_g} = \frac{mk^2\alpha}{mf_g} = \frac{k^2\alpha}{f_g}$$

h is taken in such a way that the force produces a moment about the centre of mass, which is opposite in sense to the angular acceleration α.

1.4 Dynamic Analysis of Four-Link Mechanisms

For dynamic analysis of four-link mechanisms, the following procedure may be adopted:

1. Draw the velocity and acceleration diagrams of the mechanism from the configuration diagram by usual methods.
2. Determine the linear acceleration of the centres of masses of various links, and also the angular accelerations of the links.
3. Calculate the inertia forces and inertia couples from the relations $F_i = -m f_g$ and $C_i = -I_g\alpha$.
4. Replace F_i with equivalent offset inertia force to take into account F_i as well as C_i.
5. Assume equivalent offset inertia forces on the links as static forces and analyse the mechanism by any of the methods.

1.5 Dynamic Analysis of Slider-Crank Mechanisms

The steps outlined for dynamic analysis of a four-link mechanism also hold good for a slider-crank mechanism and the analysis can be carried out in exactly the same manner.

However, an analytical approach is also being described in detail in the following sections.

1.6 Velocity and Acceleration of a Piston

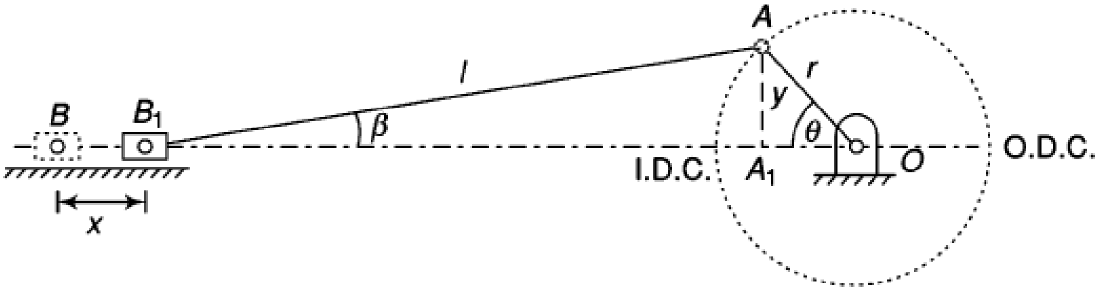

Fig.1.1 – Slider-crank mechanism

Fig.1.3 shows a slider- crank mechanism in which the crank OA rotates in the clockwise direction. *l* and *r* are the lengths of the connecting rod and the crank respectively.

Let *x* = displacement of piston from inner-dead centre At the moment when the crank has turned through angle θ from the inner-dead centre,

$$x = B_1B$$
$$= BO - B_1O$$
$$= BO - (B_1A_1 + A_1O)$$
$$= (l + r) - (l\cos\beta + \cos\theta)$$
$$= (nr + r) - (nr\cos\beta + r\cos\theta) \qquad \textit{(taking l/r = n)}$$
$$= r[(n + 1) - (n\cos\beta + \cos\theta)]$$

Where
$$\cos\beta = \sqrt{1 - sin^2\beta}$$
$$= \sqrt{1 - \frac{y^2}{l^2}}$$
$$= \sqrt{1 - \frac{(r\,sin\,\theta)^2}{l^2}}$$
$$= \sqrt{1 - \frac{sin^2\theta}{n^2}}$$
$$= \frac{1}{n}\sqrt{n^2 - sin^2\theta}$$
$$x = r[(n + 1) - \left(\sqrt{n^2 - sin^2\theta} + \cos\theta\right)]$$
$$x = r[(1 - \cos\theta) + \left(n - \sqrt{n^2 - sin^2\theta}\right)]$$

If the connecting rod is very large as compared to the crank, n^2 will be large and the maximum value of $sin^2\theta$ can be unity. Then $\sqrt{n^2 - sin^2\theta}$ approaching $\sqrt{n^2}$ or n, and

$$x = r(1 - \cos\theta)$$

This is the expression for a simple harmonic motion. Thus, the piston executes a simple harmonic motion when the connecting rod is large.

Velocity of Piston

$$v = \frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt}$$
$$= \frac{d}{d\theta}[r\{(1 - \cos\theta) + n - (n^2 - sin^2\theta)^{1/2}\}]\frac{d\theta}{dt}$$
$$= r[(0 + \sin\theta) + 0 - \frac{1}{2}(n^2 - sin^2\theta)^{\frac{1}{2}}(-2\sin\theta\cos\theta)]\omega$$
$$= r\omega\left[\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - sin^2\theta}}\right]$$

If n^2 is large compared to $sin^2\theta$,

$$= r\omega\left[\sin\theta + \frac{\sin 2\theta}{2n}\right]$$

If $\frac{\sin 2\theta}{2n}$ can be neglected (when n is quite large),

$$v = r\omega\sin\theta$$

Acceleration of Piston

$$f = \frac{dv}{dt} = \frac{dv}{d\theta}\frac{d\theta}{dt}$$

$$= \frac{d}{d\theta}\left[r\omega\left(\sin\theta + \frac{\sin 2\theta}{2n}\right)\right]\omega$$

$$= r\omega^2\left(\cos\theta + \frac{\cos 2\theta}{2n}\right)$$

If n is very very large,

$$f = r\omega^2\cos\theta \text{ as in case of SHM}$$

When θ = 0°, i.e., at IDC, $f = r\omega^2(1 + 1/n)$

When θ = 180°, i.e., at ODC, $f = r\omega^2(-1 + 1/n)$

At θ = 180°, when the direction of motion is reversed, $f = r\omega^2(1 - 1/n)$

Note that this expression of acceleration has been obtained by differentiating the approximate expression for the velocity. It is, usually, very cumbersome to differentiate the exact expression for velocity. However, this gives satisfactory results.

1.7 Angular Velocity and Angular Acceleration of Connecting Rod

$$y = l\sin\beta = r\sin\theta$$

$$\sin\beta = \frac{\sin\theta}{n}$$

Differentiating with respect to time,

$$\cos\beta\frac{d\beta}{dt} = \frac{1}{n}\cos\theta\frac{d\theta}{dt}$$

$$\frac{d\beta}{dt} = \frac{\cos\theta}{n\cos\beta}\omega$$

or

$$\omega_c = \omega\frac{\cos\theta}{n\frac{1}{n}\sqrt{n^2 - sin^2\theta}}$$

where ω_c is the angular velocity of the connecting rod

$$= \omega\frac{\cos\theta}{\sqrt{n^2 - sin^2\theta}}$$

Let α_c = angular acceleration of the connecting rod

$$\frac{d\omega_c}{dt} = \frac{d\omega_c}{d\theta}\frac{d\theta}{dt}$$

$$= \omega\frac{d}{d\theta}[\cos\theta\,(n^2 - sin^2\theta)^{-\frac{1}{2}}]\omega$$

$$= \omega^2[-\cos\theta\frac{1}{2}(n^2 - sin^2\theta)^{-\frac{3}{2}}(-2\sin\theta\cos\theta) + (n^2 - sin^2\theta)^{-\frac{1}{2}}(-\sin\theta)]$$

$$= \omega^2\sin\theta\left[\frac{cos^2\theta - (n^2 - sin^2\theta)}{(n^2 - sin^2\theta)^{\frac{3}{2}}}\right]$$

$$= -\omega^2 \sin\theta \left[\frac{(n^2-1)}{(n^2-sin^2\theta)^{\frac{3}{2}}}\right]$$

The negative sign indicates that the sense of angular acceleration of the rod is such that it tends to reduce the angle β. Thus, in the given case, the angular acceleration of the connecting rod is clockwise.

1.8 Engine Force Analysis

An engine is acted upon by various forces such as weight of reciprocating masses and connecting rod, gas forces, forces due to friction and inertia forces due to acceleration and retardation of engine elements, the last being dynamic in nature. In this section, the analysis is made of the forces neglecting the effect of the weight and the inertia effect of the connecting rod.

(i) Piston Effort (Effective Driving Force)

The piston effort is termed as the net or effective force applied on the piston. In reciprocating engines, the reciprocating masses accelerate during the first half of the stroke and the inertia force tends to resist the same. Thus, the net force on the piston is decreased. During the later half of the stroke, the reciprocating masses decelerate and the inertia force opposes this deceleration or acts in the direction of the applied gas pressure and thus, increases the effective force on the piston.

In a vertical engine, the weight of the reciprocating masses assists the piston during the outstroke (down stroke), thus, increasing the piston effort by an amount equal to the weight of the piston. During the instroke (upstroke), the piston effort is decreased by the same amount.

Let A_1 = area of the cover end
A_2 = area of the piston rod end
P_1 = pressure on the cover end
P_2 = pressure on the rod end
m = mass of the reciprocating parts
Force on the piston due to gas pressure, $F_P = P_1A_1 - P_2A_2$
Inertia force,

$$F_b = mf = mr\omega^2\left(\cos\theta + \frac{\cos 2\theta}{2n}\right)$$

which is in the opposite direction to that of the acceleration of the piston.
Net (effective) force on the piston, $F = F_P - F_b$
In case friction resistance F_f is also taken into account,
Force on the piston, $F = F_P - F_b - F_f$
In case of vertical engines, the weight of the piston or reciprocating parts also acts as force and thus force on the piston, $F = F_P + mg - F_b - F_f$

(ii) Force (thrust) along the Connecting Rod

Let F_C = Force in the connecting rod (*Fig.1.2*)
Then equating the horizontal components of forces,

$$F_C \times \cos\beta = F$$

$$F_C = \frac{F}{\cos\beta}$$

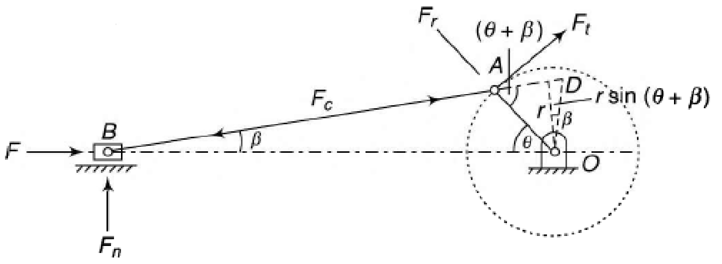

Fig.1.2

(iii) Thrust on the Sides of Cylinder

It is the normal reaction on the cylinder walls.

$$F_n = F_C \sin\beta = F \tan\beta$$

(iv) Crank Effort

Force is exerted on the crankpin as a result of the force on the piston. Crank effort is the net effort (force) applied at the crankpin perpendicular to the crank which gives the required turning moment on the crankshaft.

Let F_t = crank effort

$$F_t \times r = F_C\, r \sin(\theta + \beta)$$

$$F_t = F_C\ \sin(\theta + \beta)$$

$$= \frac{F}{\cos\beta}\sin(\theta + \beta)$$

(v) Thrust on the Bearings

The component of F_C along the crank (in the radial direction) produces a thrust on the crankshaft bearings.

$$F_r = F_C\ \cos(\theta + \beta)\frac{F}{\cos\beta}\cos(\theta + \beta)$$

1.9 Dynamically Equivalent System

The expression for the turning moment of the crankshaft has been obtained for the net force F on the piston. This force F may be the gas force with or without the consideration o f inertia force acting on the piston.

As the mass of the connecting rod is also significant, the inertia due to the same should also be taken into account. As neither the mass of the connecting rod is uniformly distributed nor the motion is linear, its inertia cannot be found as such. Usually, the inertia of the connecting rod is taken into account by considering a dynamically-equivalent system.

A dynamically equivalent system means that the rigid link is replaced by a link with two point masses in such a way that it has the same motion as the rigid link when subjected to the same force, i.e., the centre of mass of the equivalent link has the same linear acceleration and the link has the same angular acceleration.

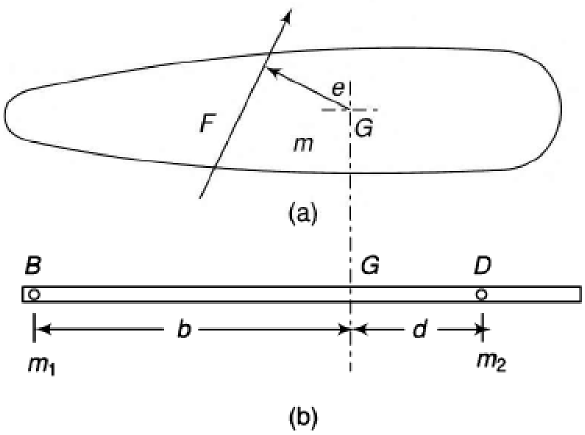

Fig.1.3

Fig.1.3 (a) shows a rigid body of mass m with the centre of mass at G. Let it be acted upon by a force F which produces linear acceleration f of the centre of mass as well as the angular acceleration of the body as the force F does not pass through G.

$$F = m.f \text{ and } F.e = I.\alpha$$

Acceleration of G, $f = {}^{F}/_{m}$

Angular acceleration of the body, $\alpha = {}^{F.e}/_{I}$

where e = perpendicular distance of *F* from G

and *I* = moment of inertia of the body about perpendicular axis through G

Now to have the dynamically equivalent system, let the replaced massless link [*Fig.1.3*(b)] has two point masses m_1 (at B and m_2 at D) at distances *b* and *d* respectively from the centre of mass G as shown in *Fig.1.3*(b).

1. To satisfy the first condition, as the force *F* is to be same, the sum of the equivalent masses m_1 and m_2 has to be equal to m to have the same acceleration. Thus, $m = m_1 + m_2$.
2. To satisfy the second condition, the numerator $F.e$ and the denominator *I* must remain the same. *F* is already taken same, Thus, e has to be same which means that the perpendicular distance of *F* from G should remain same or the combined centre of mass of the equivalent system remains at G. This is possible if

$$m_1 b = m_2 d$$

To have the same moment of inertia of the equivalent system about perpendicular axis through their combined centre of mass G, we must have

$$I = m_1 b^2 + m_2 d^2$$

Thus, any distributed mass can be replaced by two point masses to have the same dynamical properties if the following conditions are fulfilled:

(i) The sum of the two masses is equal to the total mass.

(ii) The combined centre of mass coincides with that of the rod.

(iii) The moment of inertia of two point masses about the perpendicular axis through their combined centre of mass is equal to that of the rod.

1.10 Inertia of the Connecting Rod

Let the connecting rod be replaced by an equivalent massless link with two point masses as shown in *Fig.1.4*. Let m be the total mass of the connecting rod and one of the masses be located at the small end B.

Let the second mass be placed at D and

m_b = mass at *B*

m_d = mass at *D*

Take, $BG = b$ and $DG = d$

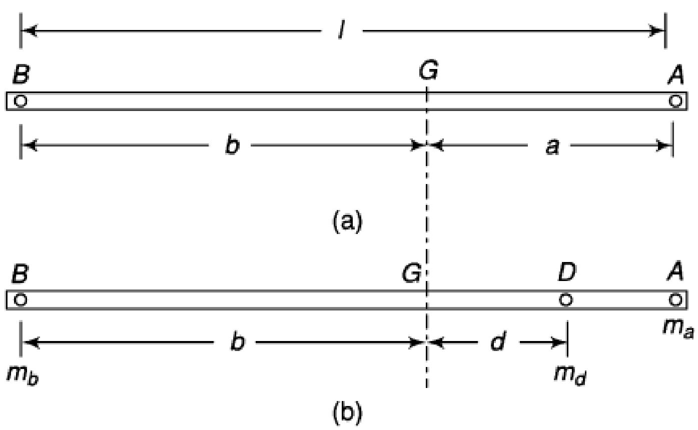

Fig.1.4

$$m_b + m_d = m \qquad \textit{Eq. (1.1)}$$

$$m_b . b = m_d . d \qquad \textit{Eq. (1.2)}$$

From *Eq. (1.1)* and *Eq. (1.2)*

$$m_b + \left(m_b \frac{b}{d}\right) = m$$

$$m_b \left(1 + \frac{b}{d}\right) = m$$

$$m_b \left(\frac{b + d}{d}\right) = m$$

$$m_b = m \frac{d}{b + d}$$

$$m_d = m \frac{b}{b + d}$$

$$I = m_b . b^2 + m_d . d^2$$

$$I = m \frac{d}{b + d} b^2 + m \frac{b}{b + d} d^2$$

$$I = mbd \left(\frac{b + d}{b + d}\right)$$

$$I = mbd$$

Let $k =$ radius of gyration of the connecting rod about an axis through the centre of mass G perpendicular to the plane of motion.

Then $$mk^2 = mbd$$

$$k^2 = bd$$

This result can be compared with that of an equivalent length of a simple pendulum in the following manner: The equivalent length of a simple pendulum is given by

$$L = \frac{k^2}{b} + b = b + d$$

where b is the distance of the point of suspension from the centre of mass of the body and k is the radius of gyration. Thus, in the present case, $b + d (= L)$ is the equivalent length if the rod is suspended from the point *B* , and *D* is the centre of oscillation or percussion.

However, in the analysis of the connecting rod, it is much more convenient if the two point masses are considered to be located at the centre o f the two end bearings, i.e., at *A* and *B*.

Let $m_a =$ mass at *A*, distance $AG = a$

Then $m_a + m_b = m$

$$m_a = m \frac{b}{a + b} = m \frac{b}{l} \qquad (l = length\ of\ rod)$$

$$m_b = m \frac{a}{a + b} = m \frac{a}{l}$$

$$I' = mab$$

Assuming $a > d,\ I' > I$

This means that by considering the two masses at *A* and *B* instead of at *D* and *B*, the inertia torque is increased from the actual value ($T = I\alpha_C$). The error is corrected by incorporating a correction couple.

Then, correction couple, $\Delta T = \alpha_C(mab - mbd)$

$$= mb\alpha_C(a - d)$$

$$= mb\alpha_C[(a + b) - (b + d)]$$

$$= mb\alpha_C(l - L) \qquad taking\ (b + d) = L$$

This correction couple must be applied in the opposite direction to that of the applied inertia torque. As the direction of the applied inertia torque is always opposite to the direction of the angular acceleration, the direction of the correction couple will be the same as that of angular acceleration, i.e., in the direction of the decreasing angle β.

The correction couple will be produced by two equal, parallel and opposite forces F_y acting at the gudgen pin and crankpin ends perpendicular to the line of stroke (*Fig.1.5*). The force at *B* is taken by the reaction of guides.

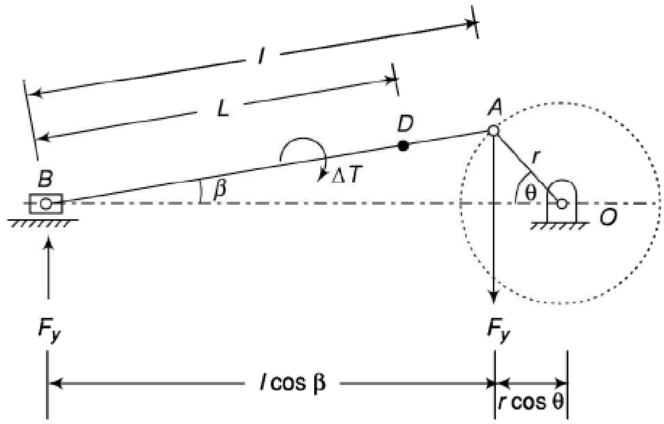

Fig.1.5

Turning moment at crankshaft due to force at A or correction torque,

$$T_C = F_y \times r\cos\theta$$

$$= \frac{\Delta T}{l\cos\beta} \times r\cos\theta \qquad (\therefore \Delta T = F_y l\cos\beta)$$

$$= \frac{\Delta T}{(l/r)}\frac{\cos\theta}{\cos\beta}$$

$$= \Delta T \frac{\cos\theta}{n\frac{1}{n}\sqrt{n^2 - sin^2\theta}}$$

$$= \Delta T \frac{\cos\theta}{\sqrt{n^2 - sin^2\theta}}$$

This correction torque is to be deducted from the inertia torque acting on the crankshaft.

Also, due to the weight of the mass at *A*, a torque is exerted on the crankshaft which is given by

$$T_a = (m_a g)r\cos\theta$$

In case of vertical engines, a torque is also exerted on the crankshaft due to the weight of mass at *B* and the expression will be similar to turning moment equation, i.e.,

$$T_b = (m_b g)r\left(\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - sin^2\theta}}\right)$$

The net torque or turning moment on the crankshaft will be the algebraic sum of the

(i) turning moment due to the force of gas pressure (T)

(ii) inertia torque due to the inertia force at the piston as a result of inertia of the reciprocating mass including the mass of the portion of the connecting rod (T_b)

(iii) inertia torque due to the weight (force) of the mass at the crank pin which is the portion of the mass of the connecting rod taken at the crank pin (T_a).

(iv) inertia torque due to the correction couple (T_C)

(v) turning moment due to the weight (force) of the piston in case of vertical engines

Usually, it is convenient to combine the forces at the piston occurring in (ii) and (v).

1.11 Inertia Force in Reciprocating Engines (Graphical Method)

The inertia forces in reciprocating engines can be obtained graphically as follows (*Fig.1.6*).

1. Draw the acceleration diagram by Klein's construction. Remember that the acceleration diagram is turned through 180° from the actual diagram and therefore, the directions of accelerations are towards O [*Fig.1.6*(a)].
2. Replace the mass of the connecting rod by a dynamically equivalent system of two masses. If one mass is placed at *B*, the other will be at *D* given by $d = k^2/b$, where k is the radius of gyration and b and d are the distances of the centre of mass from *B* and *D* respectively.

   Point *D* can also be obtained graphically. Draw $GE \perp AB$ at G and take $GE = k$. Make $\angle BED = 90°$ and obtain the point D on AB.

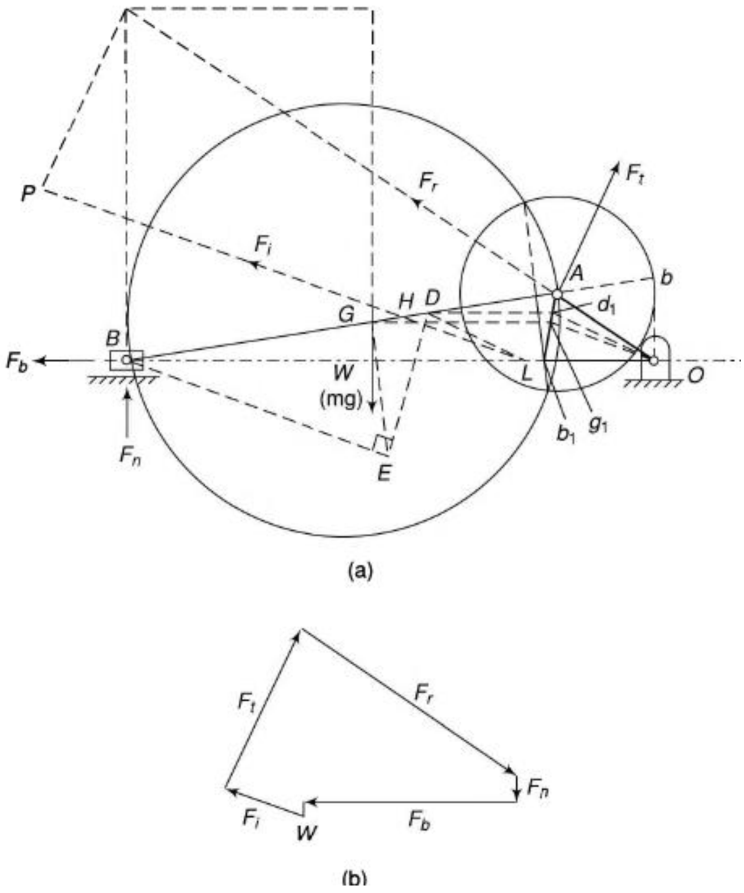

Fig.1.6

3. Obtain the accelerations of points G and D from the acceleration diagram by locating the points g_1 and d_1 on Ab_1 which represents the total acceleration of the connecting rod.

   As Ad_1/AD and Ag_1/AG are equal to Ab_1/AB, Dd_1 and Gg_1 can be drawn parallel to OB. Thus, d_1O and g_1O represent accelerations of points D and G respectively.
4. The acceleration of the mass at B is along BO and in the direction B to O. Therefore, the inertia force due to this mass acts in the opposite direction.
5. The acceleration of the mass at D is parallel to d_1O and in the direction d_1 to O, therefore, the inertia force due to this mass acts in the opposite direction through D. Draw a line parallel to Od_1 through D to represent the direction of the inertia force.

Let the lines of action of the two inertia forces due to masses at B and D meet at L. Then the resultant of the forces which is the total inertia force of the connecting rod and is parallel to Og_1 must also pass through the point L. Therefore, draw a line parallel to Og_1 through L to represent the direction of the inertia force of the connecting rod.

Now, the connecting rod is under the action of the following forces:

- Inertia force of reciprocating part F_b along OB
- The reaction of the guide F_n (magnitude and direction sense unknown)
- Inertia force of the connecting rod F_i
- The weight of the connecting rod $W(= mg)$
- Tangential force F_t at the crank pin (to be found)
- Radial force F_r at the crank pin along OA (magnitude and direction sense unknown).

Produce the lines of action of F_i and F_n to meet at I, the instantaneous centre of the connecting rod. Draw IP and IQ perpendicular to the lines of action of F_i, and the weight W respectively.

For the equilibrium of the connecting rod, taking moments about I,

$$F_t \times IA = F_b \times IB + F_i \times IP + mg \times IQ$$

Obtain the value of F_t from it and draw the force polygon to find the magnitudes and directions of forces F_r and F_n [*Fig.1.6*(b)].

In the above equation, F_t is the force required for the static equilibrium of the mechanism or it is the force required at the crank pin to overcome the inertia of the reciprocating parts and of the connecting rod. If it indicates a clockwise torque, then

Inertia torque on the crankshaft = $F_i \times OA$ counter-clockwise

1.12 References

1) Theory of Machines by S. S. Rattan, McGraw-Hill Publication.